HW #8

1. By introducing the new coordinate

$$r = \rho\big(1 + \frac{r^*}{4\rho}\big)^2,$$

show that the line element for the Schwarzschild geometry can be written in the *isotropic form*

$$ds^2 = -c^2\big(1 - \frac{r^*}{4\rho}\big)^2\big(1 + \frac{r^*}{4\rho}\big)^{-2}dt^2 + \big(1 + \frac{r^*}{4\rho}\big)^4(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2).$$

Show that $g_{00} \approx -\big(1 - \frac{r^*}{\rho}\big)$ in the weak-field limit $r^* << \rho$.

2. Show that the worldlines of radially moving photons in the Schwarzschild geometry are given by

$$ct = r + r^* \ln|\frac{r}{r^*} - 1| + \text{constant} \quad \text{(outgoing photon)},$$
$$ct = -r - r^* \ln|\frac{r}{r^*} - 1| + \text{constant} \quad \text{(incoming photon)}.$$

3. Show that, on the introduction of the advanced Eddington-Finkelstein timelike coordinate

$$ct' = ct + r^* \ln|r/r^* - 1|,$$

the Schwarzschild line element takes the form

$$ds^2 = -c^2\big(1 - \frac{r^*}{r}\big)dt'^2 + \frac{2r^*c}{r}dt'dr + \big(1 + \frac{r^*}{r}\big)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Hence, show that the worldlines of radially moving photons in advanced Eddington-Finkelstein coordinates are given by

$$ct' = r + 2r^* \ln|r/r^* - 1| + \text{constant} \quad \text{(outgoing photon)},$$
$$ct' = -r + \text{constant} \quad \text{(incoming photon)}.$$

4. Show that on introduction of the retarded Eddington-Finkelstein timelike coordinate

$$ct^* = ct - r^* \ln|r/r^* - 1|,$$

the Schwarzschild line element takes the form

$$ds^2 = -c^2\big(1 - \frac{r^*}{r}\big)dt^{*2} - \frac{2r^*c}{r}dt^*dr + \big(1 + \frac{r^*}{r}\big)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Hence, find the equations for the worldlines of radially moving photons in retarded Eddington-Finkelstein coordinates. Use this result to sketch the spacetime diagram showing the lightcone structure in this coordinate system.

(5.) A spherical distribution of dust of coordinate radius R and total mass M collapses from rest under its own gravity. Show that, as the collapse progresses, the coordinate radius r of the star's surface and the elapsed proper time τ of an observer sitting on the surface are related by

$$\tau(r) = -\frac{1}{(2GM)^{1/2}}\int_R^r \big(\frac{r}{1 - r/R}\big)^{1/2} dr.$$

By making the substitution $r = R\cos^2(\psi/2)$, or otherwise, show that the solution can be expressed parametrically as

$$r = \frac{R}{2}(1 + \cos\psi), \qquad \tau = \frac{R}{2}\big(\frac{R}{2GM}\big)^{1/2}(\psi + \sin\psi).$$

Calculate the proper time experienced by the observer before the star collapses to a point.