HW #7

In this set of homework problems we will explore various methods for deriving the equations of motion in the Schwarzschild metric and also derive expressions for the Christoffel symbols.

1. Consider the Lagrangian $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$. Use the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\mu} = 0$$

to derive the four equations of motion for a particle of mass m in the exterior region of a star of mass M. Use $r^* = 2\frac{G_N M}{c^2}$.

2. Show that a constant value of $\theta = \frac{\pi}{2}$ solves the $\mu = 2$ equation of motion from problem 1.

3. Show that the $\mu = 3$ equation corresponds to conservation of angular momentum.

Given the elements of the metric tensor, one could calculate the Christoffel symbols using the definition given in class. Alternatively, one can write the equations of motion as

$$\frac{d^2 x^\nu}{d^2\sigma} + \Gamma^\nu_{\lambda\rho}\dot{x}^\lambda \dot{x}^\rho = 0.$$

Comparing this set of equations with those obtained in problem 1 enables one to identify the non-zero Christoffel symbols very easily.

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4. Using this technique, show that

$$\begin{aligned}
\Gamma^3_{23} &= \cot\theta \\
\Gamma^3_{13} &= 1/r, \\
\Gamma^2_{12} &= 1/r, \\
\Gamma^2_{33} &= -\sin\theta\cos\theta.
\end{aligned}$$

(5.) For this problem, let $g_{00} = -e^{2\nu}$, $g_{11} = e^{2\lambda}$, and let $\nu' = d\nu/dr$, etc. Show that

$$\begin{aligned}
\Gamma^0_{10} &= \nu', \\
\Gamma^1_{00} &= \nu' e^{2(\nu-\lambda)}, \\
\Gamma^1_{11} &= \lambda', \\
\Gamma^1_{22} &= -re^{-2\lambda}, \\
\Gamma^1_{33} &= -r\sin^2\theta\; e^{-2\lambda}.
\end{aligned}$$