Mixed Integrals

1. $\displaystyle\int \frac{5-x}{2x^2+x-1}\,dx$

$$\begin{aligned}\frac{5-x}{(2x-1)(x+1)} &= \frac{A}{2x-1}+\frac{B}{x+1}\\ 5-x &= A(x+1)+B(2x-1)\end{aligned}$$

$$\begin{aligned} x &= \tfrac{1}{2} & x &= -1\\ \tfrac{3}{2}A &= \tfrac{9}{2} & -3B &= 6\\ A &= 3 & B &= -2\end{aligned}$$

$$\begin{aligned}\int \frac{5-x}{2x^2+x-1}\,dx &= \int\left(\frac{3}{2x-1}-\frac{2}{x+1}\right)\,dx\\ &= \frac{3}{2}\ln|2x-1|-2\ln|x+1|+C\end{aligned}$$

2. $\displaystyle\int x\sec x\tan x\,dx$

$$\begin{aligned} u &= x & dv &= \sec x\tan x\,dx\\ du &= dx & v &= \sec x\end{aligned}$$

$$\begin{aligned}\int x\sec x\tan x\,dx &= x\sec x-\int \sec x\,dx\\ &= x\sec x-\ln|\sec x+\tan x|+C\end{aligned}$$

3. $\displaystyle\int x\ln(x+1)\,dx$

$$\begin{aligned} u &= \ln(x+1) & dv &= x\,dx\\ du &= \frac{1}{x+1}\,dx & v &= \frac{x^2}{2}\end{aligned}$$

$$\int x\ln(x+1)\,dx = \frac{1}{2}x^2\ln(x+1)-\frac{1}{2}\int\frac{x^2}{x+1}\,dx$$

$$\begin{array}{r|rrr} \underline{-1} & 1 & 0 & 0\\ & & -1 & 1\\ \hline & 1 & -1 & 1\end{array}$$

$$\begin{aligned} &= \frac{1}{2}x^2\ln(x+1)-\frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)\,dx\\ &= \frac{1}{2}x^2\ln(x+1)-\frac{1}{2}\left(\frac{x^2}{2}-x+\ln|x+1|\right)+C\\ &= \frac{1}{2}x^2\ln(x+1)-\frac{1}{4}x^2+\frac{1}{2}x-\frac{1}{2}\ln|x+1|+C\\ &= \frac{1}{4}[2x^2\ln(x+1)-x^2+2x-2\ln|x+1|]+C\\ &= \frac{1}{4}[x^2\ln(x+1)^2-x^2+2x-\ln(x+1)^2]+C\\ &= \frac{1}{4}[(x^2-1)\ln(x+1)^2-x^2+2x]+C\end{aligned}$$

4. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^4}\ dx$

$$\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\ d\theta \\ \sqrt{1-x^2} &= \cos\theta \end{aligned}$$

$$\begin{aligned} \int \frac{\sqrt{1-x^2}}{x^4}\ dx &= \int \frac{\cos\theta}{\sin^4\theta}\ d\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\sin^2\theta}\ d\theta \\ &= \int \cot^2\theta\csc^2\theta\ d\theta \\ & u = \cot\theta \\ & du = -\csc^2\theta\ d\theta \\ &= -\int u^2\ du \\ &= -\frac{1}{3}u^3 + C \\ &= -\frac{1}{3}\cot^3\theta + C \\ &= -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C \\ &= -\frac{(1-x^2)^{3/2}}{3x^3} + C \end{aligned}$$

5. $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\ dx$

$$\begin{aligned} \frac{x^2+12x+12}{x(x-2)(x+2)} &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} \\ x^2+12x+12 &= A(x^2-4) + Bx(x+2) + Cx(x-2) \end{aligned}$$

$$\begin{aligned} x &= 0 & x &= 2 & x &= -2 \\ -4A &= 12 & 8B &= 40 & 8C &= -8 \\ A &= -3 & B &= 5 & C &= -1 \end{aligned}$$

$$\begin{aligned} \int \frac{x^2+12x+12}{x^3-4x}\ dx &= \int \left(-\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2}\right)\ dx \\ &= -3\ln|x| + 5\ln|x-2| + \ln|x+2| + C \end{aligned}$$

6. $\int \frac{\sqrt{4x^2+9}}{x^4}\ dx$

$$\int \frac{\sqrt{4x^2+9}}{x^4}\ dx = 2\int \frac{\sqrt{x^2+9/4}}{x^4}\ dx$$

$$\begin{aligned}
x &= \frac{3}{2}\tan\theta \\
dx &= \frac{3}{2}\sec^2\theta\ d\theta \\
\sqrt{x^2+9/4} &= \frac{3}{2}\sec\theta
\end{aligned}$$

$$\begin{aligned}
2\int \frac{\sqrt{x^2+9/4}}{x^4}\ dx &= 2\int \frac{\frac{3}{2}\sec\theta}{\left(\frac{3}{2}\tan\theta\right)^4}\cdot\frac{3}{2}\sec^2\theta\ d\theta \\
&= \frac{8}{9}\int \frac{\sec^3\theta}{\tan^4\theta}\ d\theta \\
&= \frac{8}{9}\int \frac{1}{\cos^3\theta}\cdot\frac{\cos^4\theta}{\sin^4\theta}\ d\theta \\
&= \frac{8}{9}\int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin^3\theta}\ d\theta \\
&= \frac{8}{9}\int \cot\theta\csc^2\theta\ d\theta \\
&= \frac{8}{9}\int \csc^2\theta\csc\theta\cot\theta\ d\theta
\end{aligned}$$

$$u = \csc\theta \quad du = -\csc\theta\cot\theta\ d\theta$$

$$\begin{aligned}
&= -\frac{8}{9}\int u^2\ du \\
&= -\frac{8}{27}u^3 + C \\
&= -\frac{8}{27}\csc^3\theta + C \\
&= -\frac{8}{27}\left(\frac{\sqrt{x^2+9/4}}{x}\right)^3 + C \\
&= -\frac{(4x^2+9)^{3/2}}{27x^3} + C
\end{aligned}$$

7. $\int \sin^3 2x\sqrt{\cos 2x}\ dx$

$$\begin{aligned}
\int \sin^3 2x\sqrt{\cos 2x}\ dx &= \sin^2 2x\sqrt{\cos 2x}\sin 2x\ dx \\
&= \int (1-\cos^2 2x)\sqrt{\cos 2x}\sin 2x\ dx
\end{aligned}$$

$$\begin{aligned}
u &= \cos 2x \\
du &= -2\sin 2x\ dx
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}\int (1-u^2)u^{1/2}\ du \\
&= -\frac{1}{2}\int (u^{1/2}-u^{5/2})\ du \\
&= -\frac{1}{2}\left(\frac{u^{3/2}}{3/2}-\frac{u^{7/2}}{7/2}\right)+C \\
&= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}-\frac{2}{7}u^{7/2}\right)+C \\
&= -\frac{1}{3}\cos^{3/2} 2x+\frac{1}{7}\cos^{7/2} 2x+C
\end{aligned}$$

8. $\int (x^2-1)e^x\ dx$

	u	v
$+$	x^2-1	e^x
$-$	$2x$	e^x
$+$	2	e^x

$$\begin{aligned}
\int (x^2-1)e^x\ dx &= (x^2-1)e^x-2xe^x+2e^x+C \\
&= x^2e^x-2xe^x+e^x+C \\
&= e^x(x^2-2x+1)+C \\
&= e^x(x-1)^2+C
\end{aligned}$$

9. $\int \tan^5 2x\sec^4 2x\ dx$

$$\begin{aligned}
\int \tan^5 2x\sec^4 2x\ dx &= \int \tan^5 2x(\tan^2 2x+1)\sec^2 2x\ dx \\
&= \int (\tan^7 2x+\tan^5 2x)\sec^2 2x\ dx
\end{aligned}$$

$$\begin{aligned}
u &= \tan 2x \\
du &= 2\sec^2 2x\ dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\int (u^7+u^5)\ du \\
&= \frac{1}{16}u^8+\frac{1}{12}u^6+C \\
&= \frac{1}{16}\tan^8 2x+\frac{1}{12}\tan^6 2x+C
\end{aligned}$$

10. $\displaystyle\int \frac{4x^2+2x-1}{x^3+x^2}\,dx$

$$\begin{aligned}\frac{4x^2+2x-1}{x^2(x+1)} &= \frac{A}{x+1}+\frac{B}{x}+\frac{C}{x^2}\\[1em] 4x^2+2x-1 &= Ax^2+Bx(x+1)+C(x+1)\\ &= Ax^2+Bx^2+Bx+Cx+C\\ &= (A+B)x^2+(B+C)x+C\end{aligned}$$

$$\begin{array}{rclrclrcl} A+B &=& 4 & B-1 &=& 2 & A+3 &=& 4\\ B+C &=& 2 & B &=& 3 & A &=& 1\\ C &=& -1 & & & & & & \end{array}$$

$$\begin{aligned}\int \frac{4x^2+2x-1}{x^3+x^2}\,dx &= \int\left(\frac{1}{x+1}+\frac{3}{x}-\frac{1}{x^2}\right)\,dx\\ &= \ln|x+1|+3\ln|x|+\frac{1}{x}+C\end{aligned}$$

11. $\displaystyle\int \frac{1}{(16-x^2)^{3/2}}\ dx$

$$\begin{aligned} x &= 4\sin\theta \\ dx &= 4\cos\theta\ d\theta \\ (\sqrt{16-x^2})^3 &= (4\cos\theta)^3 \\ &= 4^3\cos^3\theta \end{aligned}$$

$$\begin{aligned} \int \frac{1}{(16-x^2)^{3/2}}\ dx &= \int \frac{1}{4^3\cos^3\theta}\cdot 4\cos\theta\ d\theta \\ &= \frac{1}{16}\int \sec^2\theta\ d\theta \\ &= \frac{1}{16}\tan\theta + C \\ &= \frac{x}{16\sqrt{16-x^2}} + C \end{aligned}$$

12. $\displaystyle\int \frac{\cot^3 x}{\csc x}\ dx$

$$\begin{aligned} \int \frac{\cot^3 x}{\csc x}\ dx &= \int \frac{\cos^3 x}{\sin^3 x}\cdot \sin x\ dx \\ &= \int \cos^3 x \sin^{-2} x\ dx \\ &= \int \cos^2 x \sin^{-2} x \cos x\ dx \\ &= \int (1-\sin^2 x)\sin^{-2} x\cos x\ dx \end{aligned}$$

$$\begin{aligned} u &= \sin x \\ du &= \cos x\ dx \end{aligned}$$

$$\begin{aligned} &= \int (1-u^2)u^{-2}\ du \\ &= (u^{-2}-1)\ du \\ &= -u^{-1} - u + C \\ &= -\csc x - \sin x + C \end{aligned}$$

13. $\int \arcsin x \; dx$

$$\begin{aligned} u &= \arcsin x & dv &= dx \\ du &= \frac{1}{\sqrt{1-x^2}} \; dx & v &= x \end{aligned}$$

$$\begin{aligned} \int \arcsin x \; dx &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \; dx \\ & \quad u = 1 - x^2 \\ & \quad du = -2x \; dx \\ &= x \arcsin x + \frac{1}{2} \int u^{-1/2} \; du \\ &= x \arcsin x + \frac{1}{2} \cdot \frac{u^{1/2}}{1/2} + C \\ &= x \arcsin x + \sqrt{1-x^2} + C \end{aligned}$$

14. $\int \frac{x^2-1}{x^3+x} \; dx$

$$\frac{x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\begin{aligned} x^2 - 1 &= A(x^2+1) + (Bx+C)x \\ &= Ax^2 + A + Bx^2 + Cx \\ &= (A+B)x^2 + Cx + A \end{aligned}$$

$$\begin{aligned} A + B &= 1 & -1 + B &= 1 \\ C &= 0 & B &= 2 \\ A &= -1 \end{aligned}$$

$$\begin{aligned} \int \frac{x^2-1}{x^3+x} \; dx &= \int \left(-\frac{1}{x} + \frac{2x}{x^2+1} \right) \; dx \\ &= -\ln|x| + \ln|x^2+1| + C \end{aligned}$$

15. $\int \frac{x^3}{x^4 - 2x^2 - 8} \; dx$

$$\frac{x^3}{(x^2-4)(x^2+2)} = \frac{Ax+B}{x^2-4} + \frac{Cx+D}{x^2+2}$$

$$\begin{aligned} x^3 &= (Ax+B)(x^2+2) + (Cx+D)(x^2-4) \\ &= Ax^3 + 2Ax + Bx^2 + 2B + Cx^3 - 4Cx + Dx^2 - 4D \\ &= (A+C)x^3 + (B+D)x^2 + (2A-4C)x + (2B-D) \end{aligned}$$

$$\begin{array}{rcl|rcl|rcl} A + C &=& 1 & 4A + 4C &=& 4 & 4B + 4D &=& 0 \\ B + D &=& 0 & 2A - 4C &=& 0 & 2B - 4D &=& 0 \\ \hline 2A - 4C &=& 0 & 6A &=& 4 & 6B &=& 0 \\ 2B - 4D &=& 0 & A &=& \frac{2}{3} & B &=& 0 \\ & & & \frac{2}{3} + C &=& 1 & D &=& 0 \\ & & & C &=& \frac{1}{3} & & & \end{array}$$

$$\int \frac{x^3}{x^4-2x^2-8}\,dx = \int \left(\frac{2}{3}\cdot\frac{x}{x^2-4}+\frac{1}{3}\cdot\frac{x}{x^2+2}\right)\,dx$$

$$\begin{aligned} u &= x^2-4 & v &= x^2+2 \\ du &= 2x\,dx & dv &= 2x\,dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3}\int \frac{1}{u}\,du + \frac{1}{6}\int \frac{1}{v}\,dv \\ &= \frac{1}{3}\ln|x^2-4| + \frac{1}{6}\ln|x^2+2| + C \end{aligned}$$

16. $\int \frac{x^2}{\sqrt{2x-x^2}}\,dx$

$$\begin{aligned} 2x-x^2 &= -(x^2-2x+1)+1 \\ &= 1-(x-1)^2 \\ \int \frac{x^2}{\sqrt{2x-x^2}}\,dx &= \int \frac{x^2}{\sqrt{1-(x-1)^2}}\,dx \end{aligned}$$

$$\begin{aligned} x-1 &= \sin\theta \\ x &= \sin\theta+1 \\ dx &= \cos\theta\,d\theta \\ \sqrt{1-(x-1)^2} &= \cos\theta \end{aligned}$$

$$\begin{aligned} \int \frac{x^2}{\sqrt{1-(x-1)^2}}\,dx &= \int \frac{(\sin\theta+1)^2}{\cos\theta}\cdot\cos\theta\,d\theta \\ &= \int (\sin^2\theta+2\sin\theta+1)\,d\theta \\ &= \frac{1}{2}\int (1-\cos 2\theta)\,d\theta - 2\cos\theta+\theta+C \\ &= \frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right) - 2\cos\theta+\theta+C \\ &= \frac{1}{2}-\frac{1}{2}\sin\theta\cos\theta-2\cos\theta+\theta+C \\ &= \frac{3}{2}\theta-\frac{1}{2}\sin\theta\cos\theta-2\cos\theta+C \\ &= \frac{3}{2}\theta-\frac{1}{2}\cos\theta(\sin\theta-4) \\ &= \frac{3}{2}\arcsin(x-1)-\frac{1}{2}\sqrt{2x-x^2}(x-5) \\ &= \frac{3}{2}\arcsin(x-1)-\frac{1}{2}(x-5)\sqrt{2x-x^2} \end{aligned}$$

17. $\int e^{-x} \sin 2x \ dx$

$$\begin{array}{rclrcl} u & = & \sin 2x & dv & = & e^{-x}\ dx \\ du & = & 2\cos 2x\ dx & v & = & -e^{-x} \end{array}$$

$$\begin{aligned} \int e^{-x}\sin 2x &= -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\ dx \\ & \quad \begin{array}{rclrcl} u & = & \cos 2x & dv & = & e^{-x}\ dx \\ du & = & -2\sin 2x\ dx & v & = & -e^{-x} \end{array} \\ &= -e^{-x}\sin 2x - 2e^{-x}\cos 2x - 4\int e^{-x}\sin 2x\ dx \\ 5\int e^{-x}\sin 2x &= -e^{-x}(\sin 2x + 2\cos 2x) + C \\ \int e^{-x}\sin 2x &= -\frac{e^{-x}}{5}(\sin 2x + 2\cos 2x) + C \end{aligned}$$

18. $\int \sin\sqrt{x}\ dx$

$$\begin{aligned} t &= \sqrt{x} \\ dt &= \frac{1}{2\sqrt{x}}\ dx \\ &= \frac{1}{2t}\ dx \\ 2t\ dt &= dx \end{aligned}$$

$$\begin{aligned} \int \sin\sqrt{x}\ dx &= \int 2t\sin t\ dt \\ & \quad \begin{array}{rclrcl} u & = & 2t & dv & = & \sin t\ dt \\ du & = & 2\ dt & v & = & -\cos t \end{array} \\ &= -2t\cos t + 2\int \cos t\ dt \\ &= -2t\cos t + 2\sin t + C \\ &= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C \end{aligned}$$

19. $\int \sin(-4x)\cos 3x\ dx$

$$\begin{aligned} \int \sin(-4x)\cos 3x\ dx &= -\int \sin 4x\cos 3x\ dx \\ &= -\frac{1}{2}\int (\sin(4x+3x) + \sin(4x-3x))\ dx \\ &= -\frac{1}{2}\int (\sin 7x + \sin x)\ dx \\ &= -\frac{1}{2}\left(-\frac{1}{7}\cos 7x - \cos x\right) + C \\ &= \frac{1}{14}\cos 7x + \frac{1}{2}\cos x + C \end{aligned}$$

20. $\int \frac{\sqrt{x^2-25}}{x}\ dx$

$$\begin{aligned} x &= 5\sec\theta \\ dx &= 5\sec\theta\tan\theta\ d\theta \\ \sqrt{x^2-25} &= 5\tan\theta \end{aligned}$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-25}}{x}\ dx &= \int \frac{5\tan\theta}{5\sec\theta}\cdot 5\sec\theta\tan\theta\ d\theta \\
&= 5\int \tan^2\theta\ d\theta \\
&= 5\int (\sec^2\theta - 1)\ d\theta \\
&= 5(\tan\theta - \theta) + C \\
&= 5\left(\frac{\sqrt{x^2-25}}{5} + \text{arcsec}\,\frac{x}{5}\right) + C \\
&= \sqrt{x^2-25} + 5\text{arcsec}\,\frac{x}{5} + C
\end{aligned}$$

21. $\int \frac{x^2+x+3}{x^4+6x^2+9}\ dx$

$$\begin{aligned}
\frac{x^2+x+3}{(x^2+3)^2} &= \frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2} \\
x^2+x+3 &= (Ax+B)(x^2+3) + Cx + D \\
&= Ax^3 + 3Ax + Bx^2 + 3B + Cx + D \\
&= Ax^3 + Bx^2 + (3A+C)x + (3B+D)
\end{aligned}$$

$$A = 0 \qquad B = 1 \qquad C = 1 \qquad \begin{aligned} 3+D &= 3 \\ D &= 0 \end{aligned}$$

$$\int \frac{x^2+x+3}{x^4+6x^2+9}\ dx = \int \left(\frac{1}{x^2+3} + \frac{x}{(x^2+3)^2}\right)\ dx$$

$$u = x^2+3 \quad du = 2x\ dx$$

$$\begin{aligned}
&= \int \frac{1}{x^2+3}\ dx + \frac{1}{2}\int \frac{1}{u^2}\ du \\
&= \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + \frac{1}{2}(-u^{-1}) + C \\
&= \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} - \frac{1}{2(x^2+3)} + C
\end{aligned}$$

22. $\int x\sec^2 x\ dx$

$$\begin{aligned} u &= x & dv &= \sec^2 x\ dx \\ du &= dx & v &= \tan x \end{aligned}$$

$$\begin{aligned} \int x\sec^2 x\ dx &= x\tan x - \int \tan x\ dx \\ &= x\tan x + \ln|\cos x| + C \end{aligned}$$

23. $\int \sqrt{9-25x^2}\ dx$

$$\begin{aligned} \int \sqrt{9-25x^2}\ dx &= 5\int \sqrt{\frac{9}{25}-x^2}\ dx \\ x &= \frac{3}{5}\sin\theta \\ dx &= \frac{3}{5}\cos\theta\ d\theta \\ \sqrt{\frac{9}{25}-x^2} &= \frac{3}{5}\cos\theta \\ &= 5\int \frac{3}{5}\cos\theta\cdot\frac{3}{5}\cos\theta\ d\theta \\ &= \frac{9}{5}\int \cos^2\theta\ d\theta \\ &= \frac{9}{10}\int (1+\cos 2\theta)\ d\theta \\ &= \frac{9}{10}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C \\ &= \frac{9}{10}\left[\theta + \sin\theta\cos\theta\right] + C \\ &= \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x}{9}\sqrt{9-25x^2}\right] + C \\ &= \frac{9}{10}\arcsin\frac{5x}{3} + \frac{x}{2}\sqrt{9-25x^2} + C \end{aligned}$$

24. $\int \sec^3 x\ dx$

$$\begin{aligned} \int \sec^3 x\ dx &= \int \sec^2 x\sec x\ dx \\ & \quad u = \sec x \qquad dv = \sec^2 x\ dx \\ & \quad du = \sec x\tan x\ dx \qquad v = \tan x \\ &= \sec x\tan x - \int \tan^2 x\sec x\ dx \\ &= \sec x\tan x - \int (\sec^2 x - 1)\sec x\ dx \\ &= \sec x\tan x - \int \sec^3 x\ dx + \int \sec x\ dx \\ 2\int \sec^3 x\ dx &= \sec x\tan x + \int \sec x\ dx \\ \int \sec^3 x\ dx &= \frac{1}{2}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C \end{aligned}$$

25. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\, dx.$

$$
\begin{aligned}
F(x) &= \int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\, dx \\
&\quad u = \tan x \quad du = \sec^2 x\ dx \\
&= \int \frac{1}{u^2 + 5u + 6}\, du
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{(u+2)(u+3)} &= \frac{A}{u+2} + \frac{B}{u+3} \\
1 &= A(u+3) + B(u+2)
\end{aligned}
$$

$$
\begin{aligned}
u &= -2 & \quad u &= -3 \\
A &= 1 & \quad B &= -1
\end{aligned}
$$

$$
\begin{aligned}
F(x) &= \int \left(\frac{1}{u+2} - \frac{1}{u+3} \right) du \\
&= \ln|u+2| - \ln|u+3| + C \\
&= \ln|\tan x + 2| - \ln|\tan x + 3| + C
\end{aligned}
$$

26. $\int \cos(\ln x)\ dx$

$$\begin{aligned} u &= \cos(\ln x) & dv &= dx \\ du &= -\frac{\sin(\ln x)}{x}\ dx & v &= x \end{aligned}$$

$$\begin{aligned}
\int \cos(\ln x)\ dx &= x\cos(\ln x) + \int \sin(\ln x)\ dx
\end{aligned}$$

$$\begin{aligned} u &= \cos(\ln x) & dv &= dx \\ du &= -\frac{\sin(\ln x)}{x}\ dx & v &= x \end{aligned}$$

$$\begin{aligned}
&= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\ dx \\
2\int \cos(\ln(x))\ dx &= x\cos(\ln x) + x\sin(\ln x) + C \\
\int \cos(\ln x)\ dx &= \frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C
\end{aligned}$$

27. $\int \sin^2 x \cos^4 x\ dx$

$$\begin{aligned}
\int \sin^2 x\cos^4 x\ dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2\ dx \\
&= \frac{1}{8}\int (1-\cos 2x)(1+\cos 2x)(1+\cos 2x)\ dx \\
&= \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)\ dx \\
&= \frac{1}{8}\int \sin^2 2x(1+\cos 2x)\ dx \\
&= \frac{1}{8}\int (\sin^2 2x + \sin^2 2x\cos 2x)\ dx \\
&= \frac{1}{8}\int \sin^2 2x\ dx + \frac{1}{8}\int \sin^2 2x\cos 2x\ dx
\end{aligned}$$

$$u = \sin 2x$$
$$du = 2\cos 2x\ dx$$

$$\begin{aligned}
&= \frac{1}{8}\int \frac{1-\cos 4x}{2}\ dx + \frac{1}{16}\int u^2\ du \\
&= \frac{1}{16}\int (1-\cos 4x)\ dx + \frac{1}{16}\cdot\frac{1}{3}u^3 + C \\
&= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C
\end{aligned}$$

28. $\int \sqrt{x^2-4}\ dx$

$$\begin{aligned}
x &= 2\sec\theta \\
dx &= 2\sec\theta\tan\theta\ d\theta \\
\sqrt{x^2-4} &= 2\tan\theta
\end{aligned}$$

$$
\begin{aligned}
\int \sqrt{x^2-4}\;dx &= 4\int \sec\theta\tan^2\theta\;d\theta \\
&= 4\int \sec\theta(\sec^2\theta-1)\;d\theta \\
&= 4\int (\sec^3\theta-\sec\theta)\;d\theta \\
&= 4\left[\frac{1}{2}(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|)-\ln|\sec\theta+\tan\theta|\right]+C \\
&= 2(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|)-4\ln|\sec\theta+\tan\theta|+C \\
&= 2\sec\theta\tan\theta-2\ln|\sec\theta+\tan\theta|+C \\
&= 2\cdot\frac{x}{2}\cdot\frac{\sqrt{x^2-4}}{2}-2\ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|+C \\
&= \frac{1}{2}x\sqrt{x^2-4}-2\ln|x+\sqrt{x^2-4}|+C
\end{aligned}
$$

29. $\int \arctan x\;dx$

$$
\begin{aligned}
u &= \arctan x & dv &= dx \\
du &= \frac{1}{x^2+1}\;dx & v &= x
\end{aligned}
$$

$$
\begin{aligned}
\int \arctan x\;dx &= x\arctan x-\int \frac{x}{x^2+1}\;dx \\
&= x\arctan x-\frac{1}{2}\ln|x^2+1|+C \\
&= x\arctan x-\ln\sqrt{x^2+1}+C
\end{aligned}
$$

30. $\displaystyle\int \frac{1}{x\sqrt{\ln^2 x + 2}}\ dx$

$$\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\ dx \end{aligned}$$

$$\begin{aligned} \int \frac{1}{x\sqrt{\ln^2 x+2}}\ dx &= \int \frac{1}{\sqrt{u^2+2}}\ du \end{aligned}$$

$$\begin{aligned} u &= \sqrt{2}\tan\theta \\ du &= \sqrt{2}\sec^2\theta\ d\theta \\ \sqrt{u^2+2} &= \sqrt{2}\sec\theta \end{aligned}$$

$$\begin{aligned} &= \int \frac{\sqrt{2}\sec^2\theta}{\sqrt{2}\sec\theta}\ d\theta \\ &= \int \sec\theta\ d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{\sqrt{u^2+2}}{\sqrt{2}} + \frac{u}{\sqrt{2}}\right| + C \\ &= \ln|\sqrt{\ln^2 x + 2} + \ln x| + C \end{aligned}$$