# QUANTUM MECHANICS A (PHY–5645)

## HOMEWORK 8

(October 24, 2016)

Due on Tuesday, November 1, 2016

## PROBLEM 22

The Hamiltonian for a particle having an intrinsic spin of $1\hbar$ is given by

$$\hat{H} = \epsilon_0 \hat{L}_x \,,$$

where $\epsilon_0$ is a positive constant and $\hat{L}_x$ is the $x$-component of the angular momentum of the particle. In the standard canonical basis, the matrix representation of the three angular momentum matrices has the by now following familiar form (see Problem 12):

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \,, \quad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \,, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \,.$$

(a) At time $t=0$ the system is prepared in the eigenstate of $\hat{L}_x$ having the largest eigenvalue. Obtain the state of the particle $|\psi(t)\rangle$ at an arbitrary time $t \geq 0$ and compute the probability that it will remain in the same eigenstate.

(b) Now imagine that at time $t=0$ the system is prepared in the eigenstate of $\hat{L}_z$ having zero eigenvalue. That is,

$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |e_2\rangle.$$

Obtain the state of the particle $|\psi(t)\rangle$ at an arbitrary time $t \geq 0$.

(c) Compute the probabilities $P_z(1;t)$, $P_z(0;t)$, and $P_z(-1;t)$ of finding the particle in an eigenstate of $\hat{L}_z$ as a function of time for $t \geq 0$. Make a plot of the three probabilities.

## PROBLEM 23

Consider a particle of mass $m$ moving in a one-dimensional linear potential of the form

$$V(x) = \alpha |x|,$$

where $\alpha$ is a positive constant. Although exactly soluble, the solution to the linear problem involves some relatively obscures "Airy" functions. In this problem you will use the variational method to estimate the exact ground state energy. Recall that the variational theorem indicates that the exact ground-state energy $E_0$ provides a lower limit to the following energy *functional*:

$$E[\varphi] \equiv \langle\varphi|\hat{H}|\varphi\rangle \geq \langle\psi_0|\hat{H}|\psi_0\rangle = E_0.$$

That is, from all possible vectors $|\varphi\rangle$ the functional attains its minimum when the exact ground-state wave function is used.

(a) Obtain an estimate of the ground-state energy of the linear potential by using a gaussian "trial" wave-function,

$$\varphi_a(x) = \left(\frac{a}{\pi}\right)^{1/4} e^{-ax^2/2}.$$

(b) Obtain an estimate of the ground-state energy of the linear potential by using an exponential "trial" wave-function,

$$\varphi_b(x) = \sqrt{b}\, e^{-b|x|}.$$

(c) Obtain an estimate of the ground-state energy of the linear potential by using a Lorentzian "trial" wave-function,

$$\varphi_c(x) = \left(\frac{2c^3}{\pi}\right)^{1/2} \frac{1}{x^2+c^2}$$

(d) Once you have found the optimal parameters $a$, $b$, and $c$, make a plot of the three trial wave-functions.

For convenience, express the energies and distances, respectively, in units of:

$$\epsilon_0 = \left(\frac{\hbar^2\alpha^2}{2m}\right)^{1/3} \quad \text{and} \quad x_0 = \left(\frac{\hbar^2}{2m\alpha}\right)^{1/3}.$$

**Note:** Pay attention in parts (a) and (c) on how the kinetic energy favors spreading the wave-functions whereas the potential energy favors concentrating them around the origin. The optimal choice emerges from a compromise between the two contributions.

## PROBLEM 24 - Ehrenfest Theorem (Shankar Chapter 6)

Although there are enormous differences between classical and quantum mechanics, they also share a few similarities. In this problem you will prove *Ehrenfest Theorem* that establishes how the expectation value of quantum mechanical operators evolve in time. **Please resist the urge to copy the solution from Shankar.**

(a) To refresh your memory let us start with the classical case. Consider a classical observable $\omega(x,p)$, where $x$ and $p$ are the coordinate and conjugate momentum of the particle, respectively. Using Hamilton's equations of motion, show that the time evolution of $\omega$ satisfies

$$\frac{d\omega}{dt} = \{\omega, H\},$$

where $H$ is the classical Hamiltonian and "{}" denotes a Poisson bracket.

(b) In quantum mechanics all classical observables are promoted to Hermitian operators. Using the time dependent Schrödinger equation, show that the expectation value of $\hat{\Omega}$ (i.e., the operator associated with $\omega$) satisfies the following time-evolution equation:

$$\frac{d\langle\Omega\rangle}{dt} = \left(\frac{-i}{\hbar}\right)\langle[\Omega, H]\rangle,$$

where $\langle\Omega\rangle = \langle\psi(t)|\hat{\Omega}|\psi(t)\rangle$ and $|\psi(t)\rangle$ satisfies the time-dependent Schrödinger equation.