QUANTUM MECHANICS A (PHY–5645)

HOMEWORK 6

(October 4, 2016)

Due on Tuesday, October 18, 2016

PROBLEM 16

The Hamiltonian for a particle having a single "spin" degree of freedom in the presence of a constant external magnetic field is given as follows:

$$\hat{H} = -\epsilon_0 \hat{\sigma}_x \,,$$

where ϵ_0 is a constant that reflects the coupling of the particle to the external magnetic field and σ_x is the x-component of the Pauli matrices. Note that in the *standard canonical basis*, namely the basis in which $\hat{\sigma}_z$ is diagonal, the three Hermitian Pauli matrices are given by

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

At time $t = 0$ the system is prepared with "spin up" in the z-direction. That is, the system is prepared in a properly normalized state that is given in the standard canonical basis by the following expression:

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |e_1\rangle.$$

(a) Obtain the state of the particle $|\psi(t)\rangle$ at an arbitrary time $t > 0$.

(b) Compute as a function of time the probability $P(\sigma_z = -1; t)$ that the spin of the particle has flipped from spin up to spin down. In particular, obtain the first time t_0 at which the initial state of the particle "flips"; namely, the first time at which $P(\sigma_z = -1; t_0) \equiv 1$.

(c) Assuming that a measurement of the z-component of the spin was made at exactly t_0, compute the state of the system $|\psi(t)\rangle$ for $t > t_0$.

(d) If now $\hat{\sigma}_y$ is measured at a time $t > t_0$, what are the possible outcomes of the measurement and what are their respective probabilities $P(\sigma_y; t)$.

PROBLEM 17: Exercise 5.2.1 Shankar (more difficult)

We have shown in class that the eigenstates of a particle confined to a one-dimensional box of length L are given by

$$\varphi_n(x; L) = \sqrt{\frac{2}{L}} \begin{cases} \cos\left(\dfrac{n\pi x}{L}\right) & \text{if } n = 1, 3, 5, \ldots \\ \sin\left(\dfrac{n\pi x}{L}\right) & \text{if } n = 2, 4, 6, \ldots \end{cases}$$

Note that eigenstates, although not the energies, are independent of the mass of the particle and that the infinite "square-well" potential is defined as

$$V(x) = \begin{cases} 0 & \text{if } |x| < L/2, \\ \infty & \text{if } |x| \geq L/2. \end{cases}$$

Consider a particle that is in the ground state of a box of length L; that is, the wave function of the particle is given by $\psi(x) = \sqrt{2/L}\cos(\pi x/L)$. Now imagine that the box expands suddenly and symmetrically to twice its size, leaving the original state undisturbed.

(a) Compute the probability of finding the the particle in any eigenstate of the new box of size $2L$. For this you will need to expand the original wave function $\psi(x)$ in terms of the eigenstates of the new box. That is,

$$|\psi\rangle = \sum_{n=1}^{\infty} |n; 2L\rangle\langle n; 2L|\psi\rangle \rightarrow \psi(x) = \sum_{n=1}^{\infty} A_n \varphi_n(x; 2L),$$

where the expansion coefficients $A_n \equiv \langle n; 2L|\psi\rangle$ are often referred to as the "probability amplitudes".

(b) Having obtained the probability amplitudes in part (a), construct the following function:

$$\psi_N(x) = \sum_{n=1}^{N} A_n \varphi_n(x; 2L),$$

where the original state $\psi(x)$ is obtained from $\psi_N(x)$ in the limit of $N \to \infty$. Make a plot using $N=1$, $N=11$, and $N=21$ to convince yourself that $\psi_N(x)$ approaches the original wave function for large N. For this part you should use $L \equiv 1$.

PROBLEM 18: Exercise 5.2.2 Shankar (almost)

(a) Show (by essentially repeating what we did in class) that for any normalized wave function $|\psi\rangle$, the following inequality holds true:

$$\langle\psi|\hat{H}|\psi\rangle \geq E_0,$$

where E_0 is the ground-state energy (i.e., the lowest energy eigenvalue). Also show that the equality holds only for the case in which $|\psi\rangle$ is the actual ground-state wave function of $\hat{H}$. **Hint:** Expand $|\psi\rangle$ in terms of the eigenstates of $\hat{H}$.

(b) We want to prove the following theorem: *Every attractive potential in one dimension has at least one bound state.* To show this, consider the attractive "square-well" potential defined as

$$V(x) = \begin{cases} -V_0 & \text{if } |x| < L/2, \\ 0 & \text{if } |x| \geq L/2, \end{cases}$$

where $V_0 > 0$ is the depth and $L > 0$ the width of the well. To show that there exists a bound state with $E < 0$, consider the following normalized variational wave function:

$$\varphi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2},$$

and calculate

$$E(\alpha) = \langle \varphi_\alpha | \hat{H} | \varphi_\alpha \rangle \quad \text{with} \quad H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x).$$

Show that $E(\alpha)$ can always be made negative by a suitable choice of α. The desired result then follows from the application of part (a).