QUANTUM MECHANICS A (PHY–5645)

HOMEWORK 5

(September 27, 2016)
Due on Tuesday, October 4, 2016

PROBLEM 13 (the rest of Exercise 4.2.1 Shankar)

Consider the following three Hermitian operators on a Hilbert space $\mathbb{V}^3(C)$:

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Consider the state

$$|\psi\rangle = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix},$$

in the L_z basis. If L_z^2 is measured in this state and a result of $+1$ is obtained, what is the state after the measurement? How probable was this result? If L_z is measured immediately afterwards, what are the possible outcomes and their respective probabilities?

(b) A particle is in a state for which the probabilities are $P(L_z\!=\!1)\!=\!1/4$, $P(L_z\!=\!0)\!=\!1/2$, $P(L_z\!=\!-1)\!=\!1/4$. Argue convincingly that the most general normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z\!=\!1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}}|L_z\!=\!0\rangle + \frac{e^{i\delta_3}}{2}|L_z\!=\!-1\rangle\,.$$

(c) It was stated earlier on that if $|\psi\rangle$ is a normalized state, then the state $e^{i\theta}|\psi\rangle$ is a *physically* equivalent normalized state. Does this mean that the different "phase factors" $e^{i\delta_n}$ multiplying the L_z eigenstates in part (b) are all irrelevant? Answer this question by computing $P(L_x\!=\!0)$.

PROBLEM 14

Consider a particle of mass m moving under the influence of a one-dimensional harmonic potential of frequency ω given by

$$V(x) = \frac{1}{2}m\omega^2 x^2\,.$$

In a few days we will show that the discrete energy states of a *quantum* oscillator are given by $\epsilon_n\!=\!(n+\frac{1}{2})\hbar\omega$ with $n\!=\!0,1,2,\ldots$ Moreover, note that the eigenfunctions with the three lowest energies (expressed in the canonical x-basis) are given by

$$\psi_0(x) = \left(\frac{1}{\pi b^2}\right)^{1/4} e^{-x^2/2b^2} \,,$$
$$\psi_1(x) = \left(\frac{4}{\pi b^2}\right)^{1/4} \left(\frac{x}{b}\right) e^{-x^2/2b^2} \,,$$
$$\psi_2(x) = \left(\frac{1}{4\pi b^2}\right)^{1/4} \left(-1 + 2\frac{x^2}{b^2}\right) e^{-x^2/2b^2} \,,$$

where $b \equiv \sqrt{\hbar/m\omega}$ is the oscillator length.

(a) For each energy level ϵ_n, obtain the classical turning points in terms of b.

(b) Verify that $\psi_0(x), \psi_1(x)$, and $\psi_2(x)$ form an *orthonormal* set. Recall that the "dot product" of two functions is given by

$$\langle f|g\rangle \equiv \int_{-\infty}^{\infty} f^{\star}(x)g(x)dx \,.$$

(c) Make a plot of the three probability distributions $P_n(x) = |\psi_n(x)|^2$ and for all three cases determine the probability of finding the particle in the classically allowed region. Recall from part (a) that each energy level has its own classical turning points.

PROBLEM 15

Consider the above wave function associated with the second excited state of the one-dimensional harmonic oscillator. That is,

$$\psi_2(x) = \left(\frac{1}{4\pi b^2}\right)^{1/4} \left(-1 + 2\frac{x^2}{b^2}\right) e^{-x^2/2b^2} \,.$$

(a) As you know, $\psi_2(x)$ represents the *probability amplitude* of finding the particle in a small interval centered at x, with $|\psi_2(x)|^2$ representing the actual probability. Obtain the probability amplitude $\psi_2(p)$ of finding the particle with momentum p. Recall that

$$\langle x|p\rangle \equiv \varphi_p(x) = \sqrt{\frac{1}{2\pi\hbar}}\, e^{ipx/\hbar} \,.$$

(b) Make a plot of both probabilities $P_2(x) = |\psi_2(x)|^2$ and $P_2(p) = |\psi_2(p)|^2$ for the following values of b: (i) $b = 0.5$, (ii) $b = 1$, and (iii) $b = 2$.

(c) Obtain the expectation (or average) value of $\langle\hat{X}\rangle$ and its uncertainty ΔX.

(d) Obtain the expectation (or average) value of $\langle\hat{P}\rangle$ and its uncertainty ΔP.

(e) Evaluate the *product* $\Delta X \cdot \Delta P$ and show that is independent of b.