QUANTUM MECHANICS A (PHY–5645)

HOMEWORK 4

(September 20, 2016)

Due on Tuesday, September 27, 2016

PROBLEM 10

Consider two particles of masses m_1 and m_2 moving in three dimensions and interacting via a potential that only depends on the magnitude of their separation. That is, the Lagrangian of the system is given by

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|)\,.$$

(a) Rewrite the Lagrangian of the system in terms of center of mass and relative coordinates defined as follows:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \quad \text{and} \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2\,.$$

(b) Using Euler-Lagrange's equations of motion, show that the velocity of the center of mass is a constant of the motion.

(c) Obtain the corresponding Hamiltonian in terms of the generalized coordinates $\mathbf{R}$ and $\mathbf{r}$ and their associate conjugate momenta. Recall that the momentum p_q conjugate to the generalized coordinate q is defined as $p_q = (\partial L/\partial \dot{q})$.

(d) Using Hamilton's equations of motion, show that the momentum of the center of mass is a constant of the motion.

PROBLEM 11

We have shown in class using Cartesian coordinates that all the components of the angular momentum "commute" (i.e., have vanishing Poisson brackets) with the Hamiltonian of a particle of mass m moving in a spherically symmetric potential $V(r)$. In this problem you will prove the same result but now using spherical coordinates.

(a) Starting with the central-force field Lagrangian

$$L = \frac{1}{2}m\Big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\Big) - V(r)\,,$$

obtain the corresponding Hamiltonian in terms of the generalized coordinates (r, θ, φ) and their associate conjugate momenta $(p_r, p_\theta, p_\varphi)$. Recall that the momentum p_q conjugate to the generalized coordinate q is defined as $p_q = (\partial L/\partial \dot{q})$.

(b) Given the angular momentum of the particle $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, express its three cartesian components (L_x, L_y, L_z) in terms of the generalized coordinates (r, θ, φ) and their associate conjugate momenta $(p_r, p_\theta, p_\varphi)$.

(c) Show that each of the three cartesian components of the angular momentum obtained in part (b) commute with the Hamiltonian obtained in part (a).

Recall that in spherical coordinates the right-handed orthonormal basis is given by:

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\ ;\\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}\,;\\
\hat{\boldsymbol{\varphi}} &= -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}\,.
\end{aligned}$$

PROBLEM 12 (part of Exercise 4.2.1 Shankar)

Consider the following three Hermitian operators on a Hilbert space $\mathbb{V}^3(C)$:

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\ ,\quad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\ ,\quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\ .$$

(a) What are the possible values that one can obtain if L_z is measured?

(b) If the system is prepared in the state in which $L_z = -1$, what are the expectation values of $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and $\Delta L_x = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2}$.

(c) Find the eigenvalues and corresponding normalized eigenstates of L_x in the L_z-basis (i.e., in the standard canonical basis).

(d) If the system is prepared in the state in which $L_z = -1$ and L_x is measured, what are the possible outcomes and their respective probabilities?