QUANTUM MECHANICS A (PHY–5645)

HOMEWORK 3

(September 12, 2016)
Due on Tuesday, September 20, 2016

PROBLEM 7: How smart is a particle?

Consider a particle of mass $m=1$ moving, from $x_1=0$ at time $t_1=0$ to $x_2=1$ at time $t_2=\pi/2$, under the influence of a one-dimensional harmonic potential of the form:

$$V(x) = \frac{1}{2}x^2 \, .$$

(a) Compute the value of the action for a linear path of the form $x(t)=A+Bt$. Adjust the values of the constants A and B so that the boundary conditions are satisfied. If any of the constants is left undetermined, adjust it so that the action is minimized.

(b) Compute the value of the action for a quadratic path of the form $x(t)=A+Bt+Ct^2$. Adjust the values of the constants A, B, and C so that the boundary conditions are satisfied. If any of the constants is left undetermined, adjust it so that the action is minimized.

(c) Compute the value of the action for the true path, which you may obtain by any method of your choice (e.g., by using Newton's laws). Check that the true path yields an action that is smaller relative to the ones obtained in parts (a) and (b).

Read "The Principle of Least Action" from Chapter 19 of *The Feynman Lectures on Physics', Vol. II.* A true jewel written by the best physicist of the second half of the twentieth century; Enjoy!

PROBLEM 8: Exercise 2.7.2 Shankar

The definition of a *Poisson* bracket between two variables $\omega(q,p)$ and $\lambda(q,p)$ is given by the following expression:

$$\{\omega,\lambda\} \equiv \sum_n \left(\frac{\partial\omega}{\partial q_n}\frac{\partial\lambda}{\partial p_n} - \frac{\partial\omega}{\partial p_n}\frac{\partial\lambda}{\partial q_n} \right) ,$$

(a) Using Hamilton's equations of motion, show that the time evolution of q_n and p_n are given by

$$\frac{dq_n}{dt} = \dot{q}_n = \{q_n, H\} \quad \text{and} \quad \frac{dp_n}{dt} = \dot{p}_n = \{p_n, H\}$$

(b) Consider a problem in two dimensions given by the following Hamiltonian

$$H(x,y,p_x,p_y) = p_x^2 + p_y^2 + ax^2 + by^2 .$$

Argue that if $a=b$, then $\{l_z, H\}=0$, where l_z is the z-component of the angular momentum. Verify this by explicit computation.

PROBLEM 9: The Electromagnetic Lagrangian and Hamiltonian

In this problem you will "obtain" both the Lagrangian and Hamiltonian for a particle of mass m and charge q moving under the influence of an electromagnetic field characterized by a scalar potential $\Phi(\mathbf{r},t)$ and a vector potential $\mathbf{A}(\mathbf{r},t)$. Recall that the electric and magnetic fields are given in terms of the scalar and vector potentials as follows:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t},\\ \mathbf{B}(\mathbf{r},t) &= \nabla\times\mathbf{A}(\mathbf{r},t).\end{aligned}$$

(a) Assuming a Lagrangian of the following form:

$$L(\mathbf{r},\mathbf{v};t) = \frac{1}{2}m\mathbf{v}^2 - q\Phi(\mathbf{r},t) + \frac{q}{c}\mathbf{v}\cdot\mathbf{A}(\mathbf{r},t),$$

show that it generates, through the use of the Euler-Lagrange equations of motion, the well known Lorentz force. That is,

$$\mathbf{F}(\mathbf{r},t) = q\mathbf{E}(\mathbf{r},t) + \frac{q}{c}\mathbf{v}\times\mathbf{B}(\mathbf{r},t).$$

(b) Obtain the *hamiltonian function* defined as follows:

$$h(\mathbf{r},\mathbf{v};t) = \sum_i \dot{x}_i\frac{\partial L}{\partial \dot{x}_i} - L(\mathbf{r},\mathbf{v};t),$$

where $\dot{x}_i \equiv v_i$. Briefly comment on the role of the vector potential in this expression.

(c) Now obtain the Hamiltonian of the system by writing the above hamiltonian function in terms of $\mathbf{r}$ and its *canonical momentum* $\mathbf{p}$. That is,

$$H(\mathbf{r},\mathbf{p};t) = \frac{\left(\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)^2}{2m} + q\Phi(\mathbf{r},t),$$

where the canonical momentum is given by

$$p_i \equiv \frac{\partial L}{\partial \dot{x}_i}.$$

(d) Using Hamilton's equations of motion, discuss briefly whether the *canonical* momentum $\mathbf{p}$ is equal to the *kinematical* momentum $m\mathbf{v}$? This is a result of critical importance that you must always remember.