QUANTUM MECHANICS A (PHY–5645)

HOMEWORK 2

(August 29, 2016)
Due on Tuesday, September 13, 2016

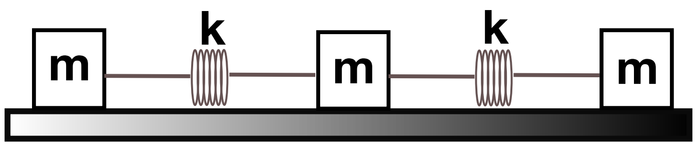
Note that for most of the problems that will be assigned during the semester, you should feel free to use Maple, Mathematica, MathLab, or any other symbolic program.

PROBLEM 4

Consider a system of three identical blocks of mass m coupled to each other by identical springs of constant k as depicted in the figure. The Lagrangian for the system is given as follows:

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - \frac{1}{2}k(x_2-x_1)^2 - \frac{1}{2}k(x_3-x_2)^2 \,,$$

where x_1, x_2, and x_3 are the displacements of the three blocks relative to their equilibrium position; in the interest of uniformity, refer to the left-most block as number 1.



(a) Using Euler-Lagrange equations of motion and Dirac's "*bra-ket*" notation, show that the system satisfies the following equation:

$$|\ddot{x}(t)\rangle + \omega_0^2\,\hat{\Omega}|x(t)\rangle = 0\,,$$

where $\omega_0 \equiv \sqrt{k/m}$ is the natural frequency of the system and $\hat{\Omega}$ is a (dimensionless) 3×3 symmetric matrix.

(b) Obtain the eigenvalues λ_n and *normalized* eigenvectors $|V_n\rangle$ of $\hat{\Omega}$. That is solve the eigenvalue equation

$$\hat{\Omega}|V_n\rangle = \lambda_n|V_n\rangle \quad (n=1,2,3)\,.$$

(c) Given that the eigenvectors of $\hat{\Omega}$ form a complete orthonormal basis, one can always write a formal solution of the Euler-Lagrange equations of motion as follows:

$$|x(t)\rangle = \sum_{n=1}^{3} |V_n\rangle\langle V_n|x(t)\rangle = \sum_{n=1}^{3} a_n(t)|V_n\rangle\,.$$

Obtain a solution for $a_n(t) \equiv \langle V_n|x(t)\rangle$ in terms of $a_n(0)$ assuming that all three blocks are initially at rest, i.e., $\dot{a}_1(0)=\dot{a}_2(0)=\dot{a}_3(0)=0$.

(d) Finish by writing the exact solution of the equations of motion in the following form:

$$|x(t)\rangle = \sum_{n=1}^{3} f_n(t)|V_n\rangle\langle V_n|x(0)\rangle,$$

making sure to clearly identify the explicit form of $f_n(t)$.

PROBLEM 5

We will continue working with the same system introduced in Problem 4. Clearly, from part (d) of Problem 4 one can write the complete solution of the equations of motion as:

$$|x(t)\rangle = \sum_{n=1}^{3} f_n(t)|V_n\rangle\langle V_n|x(0)\rangle = \hat{U}(t)|x(0)\rangle,$$

where $\hat{U}(t)$ is known as the *time evolution operator* or *propagator*, as it "propagates" the initial configuration from time $t\!=\!0$ to an arbitrary later time t.

(a) Obtain the 3×3 propagator matrix $\hat{U}(t)$.

(b) Using the expression for $\hat{U}(t)$ just derived, obtain the configuration of the system at an arbitrary time t given the following initial condition: $x_1(0)\!=\!x_2(0)\!=\!x_3(0)\!=\!1$.

(c) Repeat part (b) but now for the following initial condition: $x_1(0)=1$, $x_2(0)=-1$ and $x_3(0)\!=\!0$. Make a plot of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

PROBLEM 6: Exercise 1.8.8 Shankar

Consider 4 Hermitian matrices M_1, M_2, M_3, and M_4 that obey the following relation:

$$M_iM_j + M_jM_i \equiv \{M_i, M_j\} = 2\delta_{ij}\mathbf{1}, \quad i,j=1,\ldots 4\,,$$

where "$\{\}$" is known as the *anti-commutator* and $\mathbf{1}$ is the identity operator.

(a) Show that the eigenvalues of M_i are ±1. **Hint:** go to the eigenbasis of M_i and use the above equation for $i\!=\!j$.

(b) By considering the relation $M_iM_j\!=\!-M_jM_i$ for $i\!\neq\!j$, show that all M_i matrices are traceless. **Hint:** recall the cyclic property of the trace; $\mathrm{Tr}(ABC)\!=\!\mathrm{Tr}(CAB)$.

(c) Conclude that the matrices M_i cannot be odd-dimensional matrices.

Note that this problem is of enormous relevance to the derivation of the Dirac equation.