QUANTUM MECHANICS B (PHY–5646)

HOMEWORK 18

(March 1, 2017)
Due on Tuesday, March 7, 2017

PROBLEM 52 – (Shankar 15.3.3)

Assume without proof the validity of the following Clebsch-Gordan coefficient:

$$\langle jj|jj,10\rangle = \sqrt{\frac{j}{j+1}},$$

(a) Show that

$$\langle\alpha' j'||\hat{J}_1||\alpha j\rangle = \delta_{\alpha\alpha'}\delta_{jj'}\hbar\sqrt{j(j+1)},$$

where α (and α') represents all necessary quantum numbers required to specify the state of the system except for its angular momentum.

(b) Consider an arbitrary vector operator $\hat{\mathbf{A}}$ with cartesian components $\hat{A}_x$, $\hat{A}_y$, and $\hat{A}_z$. By defining $\hat{A}_\pm \equiv \hat{A}_x \pm i\hat{A}_y$, and using the fact that

$$\hat{\mathbf{J}}\cdot\hat{\mathbf{A}} = \frac{1}{2}\Big(\hat{J}_+\hat{A}_- + \hat{J}_-\hat{A}_+\Big) + \hat{J}_z\hat{A}_z,$$

show that

$$\langle\alpha' jm'|\hat{\mathbf{J}}\cdot\hat{\mathbf{A}}|\alpha jm\rangle = \hbar\sqrt{j(j+1)}\langle\alpha' j||\hat{A}_1||\alpha j\rangle\delta_{mm'}.$$

(c) Using the previous result, show that

$$\langle\alpha' jm'|\hat{A}_{1q}|\alpha jm\rangle = \frac{\langle\alpha' jm|\hat{\mathbf{J}}\cdot\hat{\mathbf{A}}|\alpha jm\rangle}{\hbar^2 j(j+1)}\langle jm'|\hat{J}_{1q}|jm\rangle.$$

PROBLEM 53 – (Shankar 15.3.4)

Consider a system whose total angular momentum consists of two parts $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ and whose magnetic moment is then given by

$$\hat{\boldsymbol{\mu}} = \gamma_1\frac{\hat{\mathbf{J}}_1}{\hbar} + \gamma_2\frac{\hat{\mathbf{J}}_2}{\hbar}, \quad \text{where } \gamma \equiv g\frac{q\hbar}{2mc}.$$

(a) Show using the results obtained in Problem 52 that:

1. $\langle jm|\hat{\mu}_x|jm\rangle = \langle jm|\hat{\mu}_y|jm\rangle = 0.$
2. $\langle jm|\hat{\mu}_z|jm\rangle = m\left[\frac{\gamma_1+\gamma_2}{2} + \frac{\gamma_1-\gamma_2}{2}\cdot\frac{j_1(j_1+1)-j_2(j_2+1)}{j(j+1)}\right].$

(b) Apply this result to the case of a proton (with a g-factor of $g = 5.6$) in an ${}^{2S+1}L_J = {}^2P_{1/2}$ state and show that $\langle jm|\hat{\mu}_z|jm\rangle = \pm 0.26$ nuclear magnetons. Estimate the corresponding shift in the energy of the two states if the proton is placed in a 45-Tesla magnetic field.

(c) Repeat the same calculation but now for an electron (with a g-factor of $g=2$) also in a ${}^2P_{1/2}$ state and show that $\langle jm|\hat{\mu}_z|jm\rangle = \pm 1/3$ Bohr magnetons. Estimate the corresponding shift in the energy of the two states if the electron is placed in a 45-Tesla magnetic field.

PROBLEM 54

Consider the following 2×2 Hamiltonian written in terms of the three Pauli matrices:

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_1 = \varepsilon_0 \left[\hat{\sigma}_z + \lambda \left(\hat{\sigma}_x + \hat{\sigma}_y \right) \right].$$

(a) Obtain the exact eigenvalues and normalized eigenvectors of $\hat{H}$.

(b) Compute the energies of $\hat{H}$ to second order in λ by using the following expression:

$$E_n = E_n^0 + \lambda \langle n_0|\hat{H}_1|n_0\rangle + \lambda^2 \sum_{m \neq n} \frac{|\langle n_0|\hat{H}_1|m_0\rangle|^2}{E_n^0 - E_m^0}.$$

Verify that to this order they agree with the exact expression obtained in (a).

(c) Compute the normalized eigenvectors of $\hat{H}$ to first order in λ by using the following expression:

$$|n\rangle = |n_0\rangle + \lambda \sum_{m \neq n} |m_0\rangle \frac{\langle m_0|\hat{H}_1|n_0\rangle}{E_n^0 - E_m^0}.$$

Verify that to this order they agree with the exact expression obtained in (a).