QUANTUM MECHANICS B (PHY–5646)

HOMEWORK 15

(February 1, 2017)
Due on Tuesday, February 7, 2017

PROBLEM 43

Consider the *"adjoint"* representation of SU(2) consisting of the three 3×3 matrices whose matrix elements are defined as follows:

$$[\hat{J}_i]_{jk} = -i\hbar\varepsilon_{ijk}.$$

(a) Show that the above matrices satisfy the fundamental commutation relations:

$$[\hat{J}_i, \hat{J}_j] = i\hbar\varepsilon_{ijk}\hat{J}_k.$$

(b) Show that $\hat{J}^2 \equiv \hat{J}_i\hat{J}_i$ is proportional to the 3×3 identity matrix.

(c) Obtain the eigenvalues and eigenvectors of $\hat{J}_z \equiv \hat{J}_3$.

(d) Defining the unitary matrix $\hat{U}$ as the matrix whose columns are given by the eigenvectors of $\hat{J}_z$, evaluate

$$\hat{L}_i \equiv \hat{U}^\dagger\hat{J}_i\hat{U} \quad \text{for } i = 1, 2, 3.$$

Compare the matrices $\hat{L}_i$ to the ones given on Problem 44.

Note: *For this problem assume that you are in the proverbial "deserted island" so that you only have pencil and paper available to solve this problem.*

PROBLEM 44

Hopefully in the last part of Problem 43 you obtained the following three 3×3 matrices:

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \text{and} \quad \hat{L}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

An arbitrary unit vector in real 3-dimensional space may be written as follows:

$$\hat{\mathbf{n}} = \sin(\theta)\cos(\varphi)\hat{\mathbf{x}} + \sin(\theta)\sin(\varphi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}}.$$

(a) Construct the 3×3 matrix

$$\hat{L}_n \equiv \hat{\mathbf{L}}\cdot\hat{\mathbf{n}} = \hat{L}_i n_i.$$

(b) Obtain the eigenvalues and the eigenvectors of $\hat{L}_n$.

PROBLEM 45

In this problem you will construct the eigenvectors of $\hat{J}_n$, with $\hat{\mathbf{n}}$ defined as in Problem 44, but now using the Wigner-D functions.

(a) Show that the 3-dimensional (active) rotations $\hat{R}(\varphi\hat{\mathbf{z}})\hat{R}(\theta\hat{\mathbf{y}})$ take the unit vector $\hat{\mathbf{z}}$ into the unit vector $\hat{\mathbf{n}}$. That is, $\hat{\mathbf{n}}=\hat{R}(\varphi\hat{\mathbf{z}})\hat{R}(\theta\hat{\mathbf{y}})\hat{\mathbf{z}}$ or

$$\begin{pmatrix}\sin(\theta)\cos(\varphi)\\ \sin(\theta)\sin(\varphi)\\ \cos(\theta)\end{pmatrix} = \hat{R}(\varphi\hat{\mathbf{z}})\hat{R}(\theta\hat{\mathbf{y}})\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

We have shown in class that the eigenvectors of $\hat{J}_n$ may be obtained from the eigenvector of $\hat{J}_z$ by a suitable rotation. That is,

$$|jm;\hat{\mathbf{n}}\rangle = \sum_{m'=-j}^{j} e^{-im'\varphi}\, d^j_{m',m}(\theta)|jm'\rangle,$$

where $d^j_{m',m}(\theta)$ are the *little* Wigner-D functions given in the table of Clebsch-Gordan coefficients.

(b) Obtain the three eigenstates of $\hat{J}_n$ using the relevant expressions for $d^j_{m',m}(\theta)$ for $j=1$ and compare them against the ones you obtained in Problem 44.

(c) Repeat part (b) but now for the eigenstates of $\hat{J}_n$ for the case of $j=3/2$.