QUANTUM MECHANICS B (PHY–5646)

HOMEWORK 14

(January 24, 2017)
Due on January, February 31, 2017

PROBLEM 40 – (Shankar 14.3.6)

We have shown in class that the eigenstates of the operator $\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{n}}$ are

$$|n+\rangle = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{+i\phi/2} \end{pmatrix} \quad \text{and} \quad |n-\rangle = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ +\cos(\theta/2)e^{+i\phi/2} \end{pmatrix},$$

where $\hat{\mathbf{n}}$ is a unit vector with coordinates given by $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

(a) Show by explicit calculation that, as we argued in class, $|n+\rangle$ may be obtained from the corresponding eigenstate of $\hat{\sigma}_z$ by a suitable rotation. That is,

$$|n+\rangle = \hat{U}(\phi\hat{\mathbf{z}})\hat{U}(\theta\hat{\mathbf{y}})|+\rangle.$$

(b) Show by explicit calculation that, as we argued in class, $|n-\rangle$ may be obtained from the corresponding eigenstate of σ_z by a suitable rotation. That is,

$$|n-\rangle = \hat{U}(\phi\hat{\mathbf{z}})\hat{U}(\theta\hat{\mathbf{y}})|-\rangle.$$

Note in particular the "strange" order of the rotations, i.e., first around the y-axis and then around the z-axis. We had a lot to say about this in class.

PROBLEM 41 – (Shankar 14.3.5 and 14.3.7)

Consider the 2×2 identity matrix together with the three Pauli matrices:

$$\hat{\sigma}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that $\mathrm{Tr}(\sigma_a\sigma_b) = 2\delta_{ab}$ (for $a, b = 0, x, y, z$).

(b) Express the following arbitrary 2×2 matrix

$$\hat{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

as a linear combination of $\hat{\sigma}_0 \equiv \mathbf{1}$, $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$.

(c) Express the following matrices as linear combination of $\hat{\sigma}_0 \equiv \mathbf{1}$, $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$:

1. $\hat{M}_1 = (\hat{\sigma}_0 + i\hat{\sigma}_x)^{1/2} = (\mathbf{1} + i\hat{\sigma}_x)^{1/2}$.
2. $\hat{M}_2 = (2\hat{\sigma}_0 + \hat{\sigma}_x)^{-1} = (2\mathbf{1} + \hat{\sigma}_x)^{-1}$.
3. $\hat{M}_3 = (\hat{\sigma}_x)^{-1}$.

PROBLEM 42

Consider the following Hamiltonian for the interaction between two spin-1 particles:

$$\hat{H} = \frac{\varepsilon_0}{\hbar^2}\,\hat{\mathbf{J}}(1)\cdot\hat{\mathbf{J}}(2),$$

where $\varepsilon_0 > 0$ is a positive constant that sets the energy scale for the problem.

(a) Show that the "dot" product between the angular momentum operators for the two $j=1$ particles may be re-written as follows:

$$\hat{\mathbf{J}}(1)\cdot\hat{\mathbf{J}}(2) = \frac{1}{2}\hat{J}_+(1)\hat{J}_-(2) + \frac{1}{2}\hat{J}_-(1)\hat{J}_+(2) + \hat{J}_z(1)\hat{J}_z(2),$$

where $\hat{J}_\pm$ are raising and lowering operators. We have done this in class but it is worth repeating!

(b) Using the direct-product basis for the two particles, namely,

$$|m_1\, m_2\rangle \equiv |j_1 = 1, m_1\rangle \otimes |j_2 = 1, m_2\rangle \quad (\text{with } m_1, m_2 = -1, 0, 1),$$

construct the 9×9 Hamiltonian matrix. Use the arguments developed in class to write the 9×9 Hamiltonian matrix in block-diagonal form.

(c) Using the symbolic manipulator of your choice (such as Mathematica or Maple) diagonalize the 9×9 Hamiltonian matrix to obtain all its eigenvalues and eigenvectors. Conclude that the eigenvalues of $\hat{H}$ are one-, three-, and five-fold degenerate.