QUANTUM MECHANICS B (PHY–5646)

HOMEWORK 13

(January 17, 2017)
Due on Tuesday, January 24, 2017

PROBLEM 37

Consider the fundamental commutation relations for a generalized angular momentum operators $\hat{\mathbf{J}}$. That is,

$$[\hat{J}_i, \hat{J}_j] = i\hbar\varepsilon_{ijk}\hat{J}_k.$$

We have shown in class that by virtue of the commutation relations and nothing else, $\hat{J}^2$ and $\hat{J}_z$ share the following common basis:

$$\hat{J}^2|jm\rangle = j(j+1)\hbar^2|jm\rangle,$$
$$\hat{J}_z|jm\rangle = m\hbar|jm\rangle,$$

where $j=0, 1/2, 1, 3/2, 2, \ldots$ and $m=-j, -j+1, \ldots, j-1, j$.

(a) Show that the action of the raising and lowering operators on an arbitrary state $|jm\rangle$ is given by

$$\hat{J}_+|jm\rangle = \sqrt{(j-m)(j+m+1)}\hbar\,|j, m+1\rangle,$$
$$\hat{J}_-|jm\rangle = \sqrt{(j+m)(j-m+1)}\hbar\,|j, m-1\rangle.$$

(b) By using the relations obtained in part (a) for the case of $j=3/2$, construct the three 4×4 matrices for $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$.

(c) Show explicitly that the 4×4 matrices obtained in part (b) satisfy—as they should—the fundamental commutation relations.

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The smallest set of matrices that satisfy the fundamental commutation relations are the three 2×2 Pauli matrices that are defined as follows:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that the Pauli matrices indeed satisfy

$$\left[\frac{\hat{\sigma}_i}{2}, \frac{\hat{\sigma}_j}{2}\right] = i\varepsilon_{ijk}\frac{\hat{\sigma}_k}{2}.$$

(b) Although **not** a general property of the angular momentum operators, show that the Pauli matrices also satisfy the following *anti*-commutation relations:

$$\left\{\hat{\sigma}_i, \hat{\sigma}_j\right\} \equiv \left(\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i\right) = 2\delta_{ij}.$$

(c) Use the results obtained in parts (a) and (b) to show that for two arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$ the following identity holds true:

$$\left(\hat{\boldsymbol{\sigma}}\cdot\mathbf{A}\right)\left(\hat{\boldsymbol{\sigma}}\cdot\mathbf{B}\right)=\mathbf{A}\cdot\mathbf{B}+i\hat{\boldsymbol{\sigma}}\cdot(\mathbf{A}\times\mathbf{B}).$$

(d) Use the results of part (c) to show that a finite rotation by an angle θ around an arbitrary unit vector $\hat{\mathbf{n}}$ of the form

$$\hat{U}[R(\theta\hat{\mathbf{n}})]\equiv\exp\left(-i\frac{\theta}{2}\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{n}}\right).$$

can be evaluated *exactly* to be

$$\hat{U}[R(\theta\hat{\mathbf{n}})]=\cos\left(\frac{\theta}{2}\right)-i\sin\left(\frac{\theta}{2}\right)(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{n}}).$$

In particular, conclude that the rotation matrix is not equal to the identity matrix after a $\theta=2\pi$ rotation. What is the minimum rotation angle (other than zero!) that is required for the rotation matrix to return to the identity?

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The next set of matrices that satisfy the fundamental commutation relations are the following three 3×3 matrices (for simplicity we set $\hbar\equiv1$):

$$\hat{L}_x=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad \hat{L}_y=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix},\quad \hat{L}_z=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

(a) Show that the three 3×3 matrices indeed satisfy the fundamental commutation relations

$$[\hat{L}_i,\hat{L}_j]=i\varepsilon_{ijk}\hat{L}_k.$$

(b) Show that all three matrices satisfy $\hat{L}_i^3=\hat{L}_i$ or equivalently $\hat{L}_i^3-\hat{L}_i=\hat{L}_i(\hat{L}_i^2-1)=0$. Yet, neither $\hat{L}_i=0$ nor $\hat{L}_i^2=1$; why?

(c) Use the result obtained in part (b) to show that a finite rotation by an angle θ around the x-axis can be evaluated exactly in terms of the 3×3 identity matrix $\mathbf{1}$, $\hat{L}_x$, and $\hat{L}_x^2$. That is, obtain the form of the functions $A_1(\theta)$ and $A_2(\theta)$ such that

$$\hat{U}[R(\theta\hat{\mathbf{x}})]\equiv\exp\left(-i\theta\hat{L}_x\right)=\mathbf{1}+A_1(\theta)\hat{L}_x+A_2(\theta)\hat{L}_x^2.$$