ELECTRICITY AND MAGNETISM II
Homework set #13: Electromagnetic Induction
Due on Tuesday, January 31, 2017

Problem # 13.1 :

A long solenoid with radius a and n turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\phi}$ direction. Find the electric field (magnitude and direction) at a distance s from the axis (both inside and outside the solenoid), in the quasistatic approximation.

$$\vec{B}_{\text{inside}} = \mu_0 I(t) N \hat{\phi}, \quad \vec{B}_{\text{outside}} = 0$$

For the case inside the solenoid

$$\begin{aligned}
\Phi &= \oint B \cdot \mathrm{d}a \\
&= B\pi s^2 = \mu_0 \pi s^2 I_{enc} \\
\oint \vec{E} \cdot \mathrm{d}\vec{\ell} &= E 2\pi s = -\frac{\partial \Phi}{\partial t} \\
E_{inside} &= -\frac{\mu_0 s N}{2} \frac{\partial I}{\partial t} \hat{\phi}
\end{aligned}$$

For the case outside the solenoid:

$$\begin{aligned}
\oint \vec{E} \cdot \mathrm{d}\vec{\ell} &= E 2\pi s = -\frac{\partial \Phi}{\partial t} \\
E_{outside} &= -\frac{\mu_0 a^2 N}{2s} \frac{\partial I}{\partial t} \hat{\phi}
\end{aligned}$$

Problem # 13.2 :

A long solenoid of radius a, carrying n turns per unit length, is looped by a wire with resistance R, as shown in the figure.

(a) If the current in the solenoid is increasing at a constant rate ($\mathrm{d}I/\mathrm{d}t = k$), what current flows in the loop, and which way (left or right) does it pass through the resistor?

$$\begin{aligned} B &= \mu_0 I n \\ \Phi &= \oint \vec{B} \cdot \mathrm{d}\vec{a} = \mu_0 I n \pi a^2 \\ \mathcal{E} &= -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\ &= -\mu_o \pi a^2 n \frac{\mathrm{d}I}{\mathrm{d}t} \end{aligned}$$

$$\boxed{I_{\text{loop}} = -\frac{\mu_o \pi a^2 N k}{R}} \tag{1}$$

The current passes through the resistor from left to right or counterclockwise to the current in the solenoid.

(b) If the current I in the solenoid is constant but the solenoid is pulled out of the loop (toward the left, to a place far from the loop), what total charge passes through the resistor?

$$\begin{aligned} \Delta\Phi &= \Phi(t_f) - \Phi(t_i) \\ &= -B\pi a^2 - 0 = -\pi \mu_0 n a^2 I \\ I_{\text{loop}} &= \frac{\Delta Q}{\Delta t} = \frac{\mathcal{E}}{R} \\ &= -\frac{1}{R}\frac{\Delta\Phi}{\Delta t} = \frac{\pi\mu_0 n a^2 I}{R\Delta t} \end{aligned}$$

$$\boxed{\Delta Q = \frac{\pi\mu_0 n a^2 I}{R}} \tag{2}$$

Problem # 13.3 :

A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I (see figure). Now someone cuts the wire, so I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, turn the current down gradually:

$$I(t) = \begin{cases} (1-\alpha t)I, & \text{for } 0 \le t \le 1/\alpha, \\ 0, & \text{for } t > 1/\alpha. \end{cases}$$

$$
\begin{aligned}
\Phi &= \oiint \vec{B} \cdot \mathrm{d}\vec{A} \\
&= \frac{\mu_0 I}{2\pi} \oint_s^{s+a} \frac{a\mathrm{d}s}{s} \\
&= \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s+a}{s}\right) \\
\mathcal{E} &= -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
&= -\frac{\mu_0 a}{2\pi} \ln\left(\frac{s+a}{s}\right) \frac{\mathrm{d}I}{\mathrm{d}t} \\
\mathrm{d}Q &= -\frac{\mu_0 a}{2\pi R} \ln\left(\frac{s+a}{s}\right) \mathrm{d}I \\
Q &= \frac{\mu_0 a I}{2\pi R} \ln\left(\frac{s+a}{s}\right)
\end{aligned}
$$

$$
\boxed{Q = \frac{\mu_0 a I}{2\pi R} \ln\left(\frac{s+a}{s}\right)} \tag{3}
$$

The induced current in the square loop flows counterclockwise.

Problem # 13.4 :

A small loop of wire (radius a) is held a distance z above the center of a large loop (radius b), as shown in the figure. The planes of the two loops are parallel, and perpendicular to the common axis.

(a) Suppose current I flows in the big loop. Find the flux through the little loop. (The little loop is so small that you may consider the field of the big loop to be essentially constant.)

$$\begin{aligned}
B &= \frac{\mu_o I}{4\pi}\int_0^{2\pi b}\frac{\cos\theta}{b^2+z^2}\mathrm{d}\ell \\
&= \frac{\mu_o I}{4\pi}2\pi b\frac{\cos\theta}{b^2+z^2} \\
&= \frac{\mu_0 I b^2}{2(b^2+z^2)^{3/2}} \\
\Phi &= \oint \vec{B}\cdot\mathrm{d}\vec{a}
\end{aligned}$$

$$\boxed{\Phi = \frac{\pi a^2\mu_0 I b^2}{2(b^2+z^2)^{3/2}}} \tag{4}$$

(b) Suppose current I flows in the little loop. Find the flux through the big loop. (The little loop is so small that you may treat it as a magnetic dipole.)

$$\begin{aligned}
\vec{B} &= \frac{\mu_0 a^2 I}{4r^3}\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right) \\
\Phi &= \vec{B}\cdot\mathrm{d}\vec{a} \\
&= \int B_r R^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= \int_0^{2\pi}\int_{\pi-\phi}^{\pi}\frac{\mu_0 a^2 I}{4R^3}\left(2\cos\theta\right)\left(R^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi\right) \\
&= \frac{\pi\mu_0 a^2 I}{R}\int_{\pi-\phi}^{\pi}\cos\theta\sin\theta\mathrm{d}\theta \\
&= \frac{\pi\mu_0 a^2 I}{R}\int_{\pi-\phi}^{\pi}u\mathrm{d}u \\
&= \frac{\pi\mu_0 a^2 I}{R}\frac{\sin^2\theta}{2}\Bigg|_{\pi-\phi}^{\pi} \\
&= -\frac{\pi\mu_0 a^2 I}{2R}\sin^2\phi \\
&= -\frac{\pi\mu_0 a^2 I}{2R}\left(\frac{b}{R}\right)^2
\end{aligned}$$

$$\boxed{\Phi = \frac{\pi\mu_0 a^2 I b^2}{2\left(b^2+z^2\right)^{3/2}}} \tag{5}$$

(c) Find the mutual inductances, and confirm that $M_{12} = M_{21}$

$$\Phi_1 = M_{12} I_2, \quad \Phi_2 = M_{21} I_1$$

$$\boxed{M_{12} = M_{21} = \frac{\pi \mu_0 a^2 b^2}{2\left(b^2 + z^2\right)^{3/2}}} \tag{6}$$

Problem # 13.5 :

An alternating current $I(t) = I_0 \cos(\omega t)$ (amplitude 0.5 A, frequency 60 Hz) flows down a straight wire, which runs along the axis of a toroidal coil with rectangular cross section (inner radius 1 cm, outer radius 2 cm, height 1 cm, 1000 turns). The coil is connected to a 500 Ω resistor.

(a) In the quasistatic approximation, what emf is induced in the toroid? Find the current, $I_R(t)$, in the resistor.

$$\begin{aligned}
\Phi &= \frac{\mu_0 N I}{2\pi} \int_a^b \frac{h\mathrm{d}r}{r} \\
&= \frac{\mu_0 N h}{2\pi} \ln\left(\frac{b}{a}\right) I_0 \cos(\omega t) \\
\mathcal{E} &= \frac{\mu_0 \omega N h}{2\pi} \ln\left(\frac{b}{a}\right) I_0 \sin(\omega t) \\
&= \frac{(4 \cdot 10^{-7})(60)(10^3)(10^{-2})}{2\pi} \ln(2)(0.5)\sin(\omega t)
\end{aligned}$$

$$\boxed{\mathcal{E} = 2.61 \cdot 10^{-4} \sin(\omega t)\mathrm{V}} \tag{7}$$

$$\boxed{I = \frac{2.61 \cdot 10^{-7}}{500} \sin(\omega t) = 5.22 \cdot 10^{-7} \sin(\omega t)\mathrm{A}} \tag{8}$$

(b) Calculate the back emf in the coil, due to the current $I_R(t)$. What is the ratio of the amplitudes of this back emf and the "direct" emf in (a)?

$$\begin{aligned}
\mathcal{E}_{\text{back}} &= -L\frac{\mathrm{d}I}{\mathrm{d}t} \\
L &= \frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right) \\
\mathcal{E}_{\text{back}} &= -\frac{\mu_0 \omega N^2 h}{2\pi}\ln\left(\frac{b}{a}\right) I\cos(\omega t) \\
&= \frac{(4\cdot 10^{-7})(60)(10^{6})(10^{-2})}{2\pi}\ln(2)(5.22\cdot 10^{-7})\cos(\omega t)
\end{aligned}$$

$$\boxed{\mathcal{E}_{\text{back}} = -2.74\cdot 10^{-7}\cos(\omega t)\text{V}} \tag{9}$$

$$\boxed{\frac{\mathcal{E}}{\mathcal{E}_{\text{back}}} = \frac{2.74\cdot 10^{-7}}{2.61\cdot 10^{-4}} = 1.05\cdot 10^{-3} = \frac{\mu_0 \omega N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)} \tag{10}$$