QUANTUM MECHANICS B (PHY–5646)
HOMEWORK 12

(January 9, 2017)
Due on Tuesday, January 17, 2017

PROBLEM 34

We have shown in class that for a given value of l, the spherical harmonic with the largest value of m (i.e., $m \equiv l$) is given by the following expression:

$$Y_{ll}(\theta,\varphi) = (-1)^l \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} \sin^l\theta e^{il\varphi}.$$

By using the following two relations involving the lowering operator $\hat{L}_-$:

$$\hat{L}_-|lm\rangle = \sqrt{(l+m)(l-m+1)}\hbar\,|l,m-1\rangle,$$
$$\hat{L}_- = -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\mathrm{ctg}\theta\frac{\partial}{\partial\varphi}\right),$$

generate all the spherical harmonic for the case in which $l=2$.

PROBLEM 35

The wave function of a particle of mass μ moving under the influence of a spherically symmetric potential $V(r)$ is given by

$$\langle\mathbf{r}|\Psi\rangle = \Psi(\mathbf{r}) = A(x+y+3z)e^{-r/2a_0},$$

where A is a normalization constant.

(a) Is the wave function $\Psi(\mathbf{r})$ an eigenfunction of the total angular momentum $\hat{\mathbf{L}}^2$? If so, what is the total angular momentum l? If not, what are the possible values of l that one may obtain if $\hat{\mathbf{L}}^2$ is measured.

(b) Is the wave function $\Psi(\mathbf{r})$ an eigenfunction of the z-component of the angular momentum $\hat{\mathbf{L}}_z$? If so, what is the value of m? If not, what are the possible values of m that one may obtain if $\hat{\mathbf{L}}_z$ is measured and what are their corresponding probabilities?

(c) Assume that you are told that the wave function $\Psi(\mathbf{r})$ is an energy eigenfunction and that the potential $V(r)$ vanishes as $r\to\infty$. Find the energy eigenvalue E.

(d) Having found E in part (c), find the spherically symmetric potential $V(r)$.

PROBLEM 36

Consider a particle of mass μ moving in the presence of a 3-dimensional isotropic harmonic oscillator of the form

$$V(r) = \frac{1}{2}\mu\omega^2 r^2.$$

Using the rotational invariance of the problem, we have shown in class that the eigenstates of the Hamiltonian may written as follows:

$$\psi_{nlm}(\mathbf{r}) = \frac{\mathcal{U}_{nl}(r)}{r} Y_{lm}(\theta, \phi).$$

(a) Write down the effective one-dimensional Schrödinger equation satisfied by the *reduced* radial wave function $\mathcal{U}_{nl}(r)$.

(b) You are told that $\mathcal{U}_{nl}(r)$ in the *ground state* (with $n=0$ and $l=0$) is given by

$$\mathcal{U}_{00}(r) = Are^{-\alpha r^2},$$

where A is a normalization constant and α is the (constant) range parameter. Using the effective one-dimensional Schrödinger equation, compute the ground-state energy and the value of the range parameter α.

(c) You are now told that the reduced radial wave function $\mathcal{U}_{nl}(r)$ in the *first excited state* (with $n=0$ and $l=1$) is given by

$$\mathcal{U}_{01}(r) = Br^2 e^{-\alpha r^2},$$

where B is a normalization constant and α is the same range parameter obtained in part (b). Using the effective one-dimensional Schrödinger equation, compute the energy of the first excited state.

(d) Are the reduced radial wave functions $\mathcal{U}_{00}(r)$ and $\mathcal{U}_{01}(r)$ orthogonal to each other? Discuss briefly why they should be or not be orthogonal.