# QUANTUM MECHANICS B (PHY–5646)

## HOMEWORK 11

(November 22, 2016)
Due on Thursday December 1, 2016

### PROBLEM 31 - Shankar 12.3.7 (Two-dimensional Harmonic Oscillator)

Consider a particle of mass $\mu$ moving in two spatial dimensions in the presence of an isotropic harmonic oscillator of frequency $\omega$. The Hamiltonian for such a system is given by

$$\hat{H} = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2\mu} + \frac{1}{2}\mu\omega^2\Big(\hat{X}^2 + \hat{Y}^2\Big).$$

(a) Show that the Hamiltonian commutes with the $z$-component of the angular momentum, i.e., $[\hat{H}, \hat{L}_z] = 0$, and use this fact to reduce the eigenvalue problem for $\hat{H}$ to the radial differential equation for $R_{Em}(\rho)$. That is, show that $R_{Em}(\rho)$ satisfies the following differential equation:

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} + \frac{2\mu}{\hbar^2}\left(E - \frac{1}{2}\mu\omega^2\rho^2\right)\right] R_{Em}(\rho) = 0.$$

(b) As in the case of the one-dimensional harmonic oscillator, introduce the following two dimensionless variables

$$\epsilon = \frac{E}{\hbar\omega} \quad \text{and} \quad y = \frac{\rho}{b} \equiv \sqrt{\frac{\mu\omega}{\hbar}}\rho$$

to rewrite the differential equation as

$$R''_{Em}(y) + \frac{1}{y}R'_{Em}(y) + \left(2\epsilon - \frac{m^2}{y^2} - y^2\right) R_{Em}(y) = 0.$$

(c) By examining the limits as $y \to 0$ and as $y \to \infty$ one can motivate rewriting $R_{Em}$ as

$$R_{Em}(y) = y^{|m|}e^{-y^2/2}U(y).$$

Show that $U(y)$ satisfies the following differential equation:

$$U''(y) + \left(\frac{2|m|+1}{y} - 2y\right) U'(y) + \Big(2\epsilon - 2|m| - 2\Big)U(y) = 0.$$

(d) Finally, by changing variables one last time to $x = y^2$, show that $U(x)$ satisfies the following differential equation

$$xU''(x) + (\beta - x)U'(x) - \alpha U(x) = 0.$$

Make sure to express the two constants $\alpha$ and $\beta$ in terms of $\epsilon$ and $|m|$.

**PROBLEM 32 - Shankar 12.3.7 (Two-dimensional Harmonic Oscillator)**

The differential equation obtained in part (d) of Problem 34 has as a solution the *degenerate hypergeometric function (also called Kummer's confluent hypergeometric function)* $F(\alpha, \beta, x)$ that is defined in terms of an infinite power series in $x$ as follows:

$$F(\alpha, \beta, x) = 1 + \frac{\alpha}{\beta} x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} \frac{x^3}{3!} + \dots \tag{1}$$

In general, the degenerate hypergeometric function $F(\alpha, \beta, x)$ increases for $x \gg 1$ as $F(\alpha, \beta, x) \propto e^x$, so the radial component of the wave function $R_{Em}$ will *diverge* at large distances as $R_{Em}(y) \propto e^{y^2/2}$. However, if the series in Eq. (1) can be made to terminate, then $F(\alpha, \beta, x)$ reduces to a polynomial and $R_{Em}(y)$ will *decrease* at large distances, as required.

(a) Find that the values of $\alpha$ that are required to reduce $F(\alpha, \beta, x)$ to a polynomial of degree $k = 0, 1, 2, \dots$ and show that the corresponding quantized energies are given by

$$\epsilon_{km} = (2k + |m| + 1) \equiv (n+1).$$

(b) For a given value of $n$, what are the allowed values of $|m|$? Given this information, show that for a fixed value of $n$ the degeneracy is $n+1$. Compare this value of the degeneracy to the answer that you have obtained in Cartesian coordinates where the energies are given by

$$\epsilon_{n_x n_y} = (n_x + n_y + 1) \equiv (n+1).$$

(c) Write down all the normalized eigenfunctions (in polar coordinates) corresponding to $n = 0$ and $n = 1$.

(d) Argue that the $n = 0$ wave function must equal the corresponding one in Cartesian coordinates and that the $n = 1$ solutions are linear combinations of the Cartesian counterparts.

## PROBLEM 33 - Shankar 12.3.8 (Landau Levels)

Consider a particle of mass $\mu$ and charge $q$ in the presence of a vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{B}{2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})\,,$$

where $B$ is the constant magnitude of the magnetic field.

(a) Show that the magnetic field is given by $\mathbf{B} = B\hat{\mathbf{z}}$. Incidentally, show that both $\mathbf{A}'(\mathbf{r}) = -yB\hat{\mathbf{x}}$ and $\mathbf{A}''(\mathbf{r}) = xB\hat{\mathbf{y}}$ also generate the same exact magnetic field. Do you understand the significance of these equivalent vector potentials?

(b) Show that *classically* a particle in this potential will move in a "cyclotron" orbit, i.e., in a circle at the cyclotron frequency $\omega_0 = qB/\mu c$.

(c) Consider the Hamiltonian for the corresponding quantum problem:

$$\hat{H} = \frac{(\hat{P}_x + qB\hat{Y}/2c)^2}{2\mu} + \frac{(\hat{P}_y - qB\hat{X}/2c)^2}{2\mu}.$$

Show that the operators $\hat{Q} = (c\hat{P}_x + qB\hat{Y}/2)/qB$ and $\hat{P} = (\hat{P}_y - qB\hat{X}/2c)$ are canonical, i.e., they satisfy the fundamental commutation relation: $[\hat{Q}, \hat{P}] = i\hbar$.

(d) Write the Hamiltonian in terms of the transformed operators $\hat{P}$ and $\hat{Q}$ and show that the allowed energy levels are given by $E_n = (n + 1/2)\hbar\omega_0$.

(e) Expand the Hamiltonian in terms of the original operators ($\hat{P}_x$, $\hat{P}_y$, $\hat{X}$, $\hat{Y}$) and show that it can be written as follows:

$$\hat{H} = \hat{H}_0\left(\frac{\omega_0}{2}, \mu\right) - \frac{1}{2}\omega_0\hat{L}_z,$$

where $H_0$ is the Hamiltonian of an isotropic two-dimensional harmonic oscillator of mass $\mu$ and frequency $\omega_0/2$ and $\hat{L}_z$ is the $z$-component of the angular-momentum operator. Argue that the same basis that diagonalizes $\hat{H}_0$ will also diagonalize $\hat{H}$.

(f) By thinking in terms of the basis found in (e), show that allowed energy levels are given by $E_{km} = \frac{1}{2}(2k + |m| - m + 1)\hbar\omega_0$, where $k$ is any integer and $m$ is the $z$-component of the angular momentum. Convince yourself that you get the same energy levels from this formula as from the one obtained in part (d).