QUANTUM MECHANICS A (PHY–5645)

HOMEWORK 10

(November 8, 2016)
Due on Tuesday, November 22, 2016

PROBLEM 28 - Shankar Exercise 7.4.8

Consider the three components of the angular momentum in classical mechanics:

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} \quad \text{or} \quad l_j = \varepsilon_{jkn}\, x_k\, p_n = \begin{cases} yp_z - zp_y & \text{if } j = 1, \\ zp_x - xp_z & \text{if } j = 2, \\ xp_y - yp_x & \text{if } j = 3. \end{cases}$$

(a) Promote the classical angular momentum variables l_j to quantum mechanical Hermitian operators $\hat{L}_j$ and show that there are no *ordering ambiguities*.

(b) Verify that the classical angular momentum variables satisfy the following relations between Poisson brackets:

$$\{l_j, l_k\} = \varepsilon_{jkn}\, l_n.$$

(c) Using the fundamental commutation relations, *i.e.*, $[\hat{X}_j, \hat{P}_k] = i\hbar\delta_{jk}$, verify that the quantum angular momentum operators satisfy the following commutation relations:

$$[\hat{L}_j, \hat{L}_k] = i\hbar\varepsilon_{jkn}\hat{L}_n.$$

Try to solve this part using Levi-Civita symbols and the commutator identities given in page 20 of Shankar.

PROBLEM 29 - Shankar Exercise 10.2.3

The Hamiltonian for the three dimensional isotropic harmonic oscillator is given by

$$\hat{H} = \frac{1}{2m}\left(\hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2\right) + \frac{1}{2}m\omega^2\left(\hat{X}^2 + \hat{Y}^2 + \hat{Z}^2\right).$$

(a) By writing the eigenstates of the Hamiltonian in the *energy* basis as $|n\rangle = |n_x, n_y, n_z\rangle$ (or equivalently $|n\rangle = |n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle$), show that the eigenvalues of the Hamiltonian are given by

$$E(n_x, n_y, n_z) = \left(N + \frac{3}{2}\right)\hbar\omega; \quad \text{where} \quad N = n_x + n_y + n_z.$$

(b) Write the eigenfunctions of the Hamiltonian in terms of single-oscillator wave-functions in the $|x, y, z\rangle$ basis and verify that the parity of a state with a given N is $(-1)^N$. Moreover, show that the degeneracy of a level with energy $E = \left(N + \frac{3}{2}\right)\hbar\omega$ is $g = (N+1)(N+2)/2$.

(c) Re-write the first four eigenstates of the Hamiltonian (*i.e.*, the ones with the four lowest energies) in terms of spherical coordinates (r, θ, ϕ) rather than cartesian coordinates (x, y, z).

PROBLEM 30 - Shankar Exercise 10.3.3

Imagine a situation in which there are three particles and only three states $|a\rangle$, $|b\rangle$, $|c\rangle$, with energies $\epsilon_a < \epsilon_b < \epsilon_c$. You may assume that the three states form an orthonormal basis.

(a) Show that the total number of allowed, distinct configurations for this system is 27 if all three particles are *distinguishable*.

(b) Show that the total number of allowed, distinct configurations for this system is 10 if all three particles are *identical bosons*. Compute the energy of each of these 10 states and write the normalized first excited state of the system taking into account that it must be *symmetric* under the exchange of any two bosons.

(c) Show that the total number of allowed, distinct configurations for this system is 1 if all three particles are *identical fermions*. Compute the energy of this unique state and write the normalized state of the system taking into account that it must be *anti-symmetric* under the exchange of any two fermions.