Quantum Mechanics A (PHY-5645)
Homework 1
(August, 29, 2016)
Due on Thursday, September 8, 2016
[redacted]

Problem 1: Exercise 1.8.5 Shankar (almost!)

Consider the following matrix that represents a general rotation by an angle θ around the z-axis:

$$R_z(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Show that the matrix $R_z(\theta)$ is unitary; note that in the case that all the entries of the matrix are real the matrix is said to be *orthogonal.*

$$UU^\dagger = I$$

$$R_z^\dagger = \begin{pmatrix} cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

$$R_z R_z^\dagger = \begin{pmatrix} cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2)$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta cos\theta + \sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \quad (4)$$

(b) Show that the eigenvalues of $R_z(\theta)$ are given by $e^{i\theta}$, $e^{-i\theta}$, and 1.

$$R_z(\theta) = \begin{pmatrix} \cos\theta - \lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix}$$

$$\begin{aligned} det|R_z(\theta)| &= (1-\lambda)[(\cos\theta-\lambda)^2-(-\sin^2\theta)] && (5)\\ &= (1-\lambda)(\lambda^2-2\lambda\cos\theta+1) && (6)\\ 1-\lambda=0 \qquad & \lambda^2-2\lambda\cos\theta+1=0 && (7)\\ & \frac{2\cos\theta\pm\sqrt{(-2\cos\theta)^2-4}}{2} && (8)\\ & cos\theta\pm i\sin\theta = e^{i\theta}, e^{-i\theta} && (9)\\ & \lambda = 1, e^{i\theta}, e^{-i\theta} && (10)\end{aligned}$$

(c) Find the eigenvectors of $R_z(\theta)$ and show that they are orthogonal.
For $\lambda = 1$:

$$\begin{pmatrix} \cos\theta-1 & \sin\theta & 0\\ -\sin\theta & \cos\theta-1 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$x_1 = \frac{-\sin\theta}{\cos\theta-1}x_2 = \frac{\cos\theta-1}{\sin\theta}x_2 \tag{11}$$

$$X_1 = x_3\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{12}$$

For $\lambda = e^{i\theta}$:

$$\begin{pmatrix} -i\sin\theta & \sin\theta & 0\\ -\sin\theta & -i\sin\theta & 0\\ 0 & 0 & 1-e^{i\theta}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$x_1 = -ix_2 \tag{13}$$

$$X_2 = \frac{x_2}{\sqrt{2}}\begin{pmatrix}i\\1\\0\end{pmatrix} \tag{14}$$

For $\lambda = e^{-i\theta}$:

$$\begin{pmatrix} i\sin\theta & \sin\theta & 0\\ -\sin\theta & i\sin\theta & 0\\ 0 & 0 & 1-e^{-i\theta}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$x_1 = ix_2 \tag{15}$$

$$X_3 = \frac{x_2}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \tag{16}$$

Eigenvectors are orthogonal if they equal zero when multiplied:

$$\langle X_1|X_3\rangle = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \varnothing$$

$$\langle X_1|X_2\rangle = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \varnothing$$

$$\langle X_3|X_2\rangle = \begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \varnothing$$

(d) Verify that the *transformed* matrix $U^\dagger R_z(\theta) U = diag(e^{i\theta}, e^{-i\theta}, 1)$ is the diagonal matrix of eigenvalues, where U is the matrix of eigenvectors, namely, the one that has the *normalized* eigenvectors of $R_z(\theta)$ as its columns.

$$U = \begin{pmatrix} -i/\sqrt{2} & i/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} & 0 \\ -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U^\dagger R_z U = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

$$= \begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} & 0 \\ -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} ie^{i\theta} & \frac{1}{\sqrt{2}} ie^{-i\theta} & 0 \\ \frac{1}{\sqrt{2}} e^{i\theta} & \frac{1}{\sqrt{2}} e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{i\theta} & -\frac{1}{2}e^{-i\theta} + \frac{1}{2}e^{-i\theta} & 0 \\ -\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{i\theta} & \frac{1}{2}e^{-i\theta} + \frac{1}{2}e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{19}$$

Problem 2

Euler's theorem for the rotation of a rigid body effectively says that any number of successive rotations are equivalent to a single rotation by a given angle along a given direction. In this problem you are going to verify Euler's theorem by considering the following two matrices that represent rotations by an angle $\pi/2$ along the x-axis and z-axis, respectively:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a)Show that the matrix $R_{xz} = R_z \cdot Rx$ may be represented as a single rotation by explicitly finding the angle of rotation θ and the axis of rotation $\hat{\boldsymbol{n}}$.

$$R_{xz} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For $\lambda = 1$:

$$x_1 = x_2 = x_3 \qquad \hat{\boldsymbol{n}} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

A dummy vector is multiplied by the eigenvector which yields a null vector. This indicates a vector perpendicular to the axis of rotation:

$$\nu\hat{\boldsymbol{n}} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \varnothing$$

Next apply rotational matrix R_{xz} to dummy vector ν and find angle θ from dot product equations:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \nu' \tag{20}$$

$$\arccos\left[\frac{\nu \cdot \nu'}{\|\nu\|\|\nu'\|}\right] = \theta \tag{21}$$

$$\arccos\left[\frac{-1}{2}\right] = \frac{2\pi}{3} \tag{22}$$

(b)Do the same as in part (a), but now for the matrix $R_{zx} = R_x \cdot R_z$.

$$R_{zx} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

For $\lambda = 1$:

$$x_1 = -x_2 = x_3 \qquad \hat{\boldsymbol{m}} = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$$

A dummy vector is multiplied by the eigenvector to yield a null vector as in part a:

$$u\hat{\boldsymbol{m}} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \varnothing$$

rotational matrix R_{zx} is applied dummy vector u :

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = u' \tag{23}$$

$$\arccos\left[\frac{u \cdot u'}{\|u\|\|u'\|}\right] = \theta \tag{24}$$

$$\arccos\left[\frac{-1}{2}\right] = \frac{2\pi}{3} \tag{25}$$

(c) The rotation angles in part (b) are the same as the angles found in part (a). However, the axes of rotation were not the same in parts (a) and (b).

Problem 3: Exercise 1.8.10 Shankar (almost!)

Consider the following two 3 × 3 symmetric matrices:

$$\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \textbf{and} \quad \Lambda = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(a) Find the eigenvalues of both and Ω and Λ.

$$det\,|\Omega - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = \lambda\left[(1-\lambda)^2\right] = 0 \tag{26}$$

$$\lambda = 0, 2 \tag{27}$$

$$det\,|\Lambda - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & -\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} \tag{28}$$

$$= (2-\lambda)\left[(-\lambda)(2-\lambda) - 1\right] - (2-\lambda) = 0 \tag{29}$$

$$(2-\lambda)(\lambda^2 - 2\lambda - 2) = 0 \tag{30}$$

$$\lambda = 2 \pm \frac{\sqrt{(-2)^2 - 4(-2)}}{2} \tag{31}$$

$$\lambda = 2, 1 \pm \sqrt{3} \tag{32}$$

(b) Show that the commutator of Ω and Λ vanishes. This implies that they share a common set of eigenvectors.

$$[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega$$

$$\Omega\Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix} \tag{33}$$

$$\Lambda\Omega = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix} \tag{34}$$

Therefore:

$$\Omega\Lambda - \Lambda\Omega = 0$$

(c) Find the set of normalized eigenvectors common to both Ω and Λ.
Eigenvectors of Ω:
For $\lambda = 0$:

$$det|\Omega| = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} =$$ [redacted] 0

$$x_1 = -x_3 \qquad x_2 = x_2 \tag{35}$$

$$X_1 = x_1 \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{36}$$

For $\lambda = 2$:

$$det|\Omega - 2I| = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 = x_3 \qquad x_2 = 0 \tag{37}$$

$$X_2 = x_1 \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \tag{38}$$

<u>Eigenvectors of Λ:</u>

For $\lambda = 2$:

$$x_1 = x_3 \qquad x_2 = 0 \tag{39}$$

$$X_1 = x_1 \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \tag{40}$$

For $\lambda = 1 + \sqrt{3}$:

$$x_1 = -(1 - \sqrt{3}) \qquad x_2 = -x_3 \tag{41}$$

$$X_2 = \frac{x_1}{\sqrt{6 - 2\sqrt{3}}} \begin{pmatrix} 1 \\ -(1 - \sqrt{3}) \\ -1 \end{pmatrix} \tag{42}$$

For $\lambda = 1 - \sqrt{3}$:

$$x_1 = -(1 + \sqrt{3}) \qquad x_2 = -x_3 \tag{43}$$

$$X_3 = \frac{x_1}{\sqrt{6 + 2\sqrt{3}}} \begin{pmatrix} 1 \\ -(1 + \sqrt{3}) \\ -1 \end{pmatrix} \tag{44}$$

(d) Find the unitary transformation U that brings Ω and Λ into diagonal form and verify that $U^\dagger \Omega U$ and $U^\dagger \Lambda U$ are the respective diagonal matrices of eigenvalues.

For this last part you should feel free to use Maple, Mathematica, MathLab, or any other symbolic program of your choice.

For Ω:

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$\Omega U = \begin{pmatrix} 0 & 0 & 2/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{pmatrix} \tag{45}$$

$$U^\dagger \left(\Omega U\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{46}$$

For Λ:

$$U = \begin{pmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{6-2\sqrt{3}}} & \frac{1}{\sqrt{6+2\sqrt{3}}} \\ 0 & \frac{-(1-\sqrt{3})}{\sqrt{6-2\sqrt{3}}} & \frac{-(1+\sqrt{3})}{\sqrt{6+2\sqrt{3}}} \\ 1/\sqrt{2} & \frac{-1}{\sqrt{6-2\sqrt{3}}} & \frac{-1}{\sqrt{6+2\sqrt{3}}} \end{pmatrix} \quad U^\dagger = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ \frac{1}{\sqrt{6-2\sqrt{3}}} & \frac{-(1-\sqrt{3})}{\sqrt{6-2\sqrt{3}}} & \frac{-1}{\sqrt{6-2\sqrt{3}}} \\ \frac{1}{\sqrt{6+2\sqrt{3}}} & \frac{-(1+\sqrt{3})}{\sqrt{6+2\sqrt{3}}} & \frac{-1}{\sqrt{6+2\sqrt{3}}} \end{pmatrix}$$

$$U^\dagger \left(\Lambda U\right) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+\sqrt{3} & 0 \\ 0 & 0 & 1-\sqrt{3} \end{pmatrix} \tag{47}$$