QUANTUM MECHANICS B (PHY5646)
HOMEWORK 17

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PROBLEM 49

Consider three spin-1/2 particles (say an up, a down, and a strange quark).
(a) Using the law of addition of angular momentum, show that:

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$$

$$\begin{aligned}
j_1 \otimes j_2 &= (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \ldots \oplus (j_1 - j_2) \\
\left(\frac{1}{2} + \frac{1}{2} \oplus \frac{1}{2} - \frac{1}{2}\right) \otimes \frac{1}{2} &= \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} \\
(1 \oplus 0) \otimes \frac{1}{2} &= \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} \\
\left(1 \otimes \frac{1}{2}\right) \oplus \left(\frac{1}{2} \otimes 0\right) &= \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} \\
\left(\frac{3}{2} \oplus \frac{1}{2}\right) \oplus \frac{1}{2} &= \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}
\end{aligned}$$

$$\boxed{\therefore \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}} \tag{1}$$

(b) Construct using your method of choice, the eigenstates of good total angular momentum in terms of the direct product basis of the three particles.

$$\left(1 \otimes \frac{1}{2}\right) \Rightarrow |j_{12}; j_3, m\rangle$$

[redacted]

$$\begin{aligned}
\left|1 \frac{3}{2} \frac{3}{2}\right\rangle &= \left|11; \frac{1}{2} \frac{1}{2}\right\rangle = \boxed{\left|\frac{1}{2} \frac{1}{2}; \frac{1}{2} \frac{1}{2}; \frac{1}{2} \frac{1}{2}\right\rangle} \\
\left|1 \frac{3}{2} \frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}} \left|11; \frac{3}{2} \frac{-1}{2}\right\rangle + \sqrt{\frac{2}{3}} \left|10; \frac{3}{2} \frac{1}{2}\right\rangle
\end{aligned}$$

$$\left|1\frac{3}{2}\frac{1}{2}\right\rangle = \boxed{\frac{1}{\sqrt{3}}\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2}\right\rangle}$$

$$\left|1\frac{3}{2}\frac{-1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|10;\frac{1}{2}\frac{-1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|1-1;\frac{1}{2}\frac{1}{2}\right\rangle$$

$$\left|1\frac{3}{2}\frac{-1}{2}\right\rangle = \boxed{\frac{1}{\sqrt{3}}\left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2}\right\rangle}$$

$$\left|1\frac{3}{2}\frac{-3}{2}\right\rangle = \left|1-1;\frac{1}{2}\frac{-1}{2}\right\rangle = \boxed{\left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle}$$

$$\left|1\frac{1}{2}\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|11;\frac{1}{2}\frac{-1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|10;\frac{1}{2}\frac{1}{2}\right\rangle$$

$$\left|1\frac{1}{2}\frac{1}{2}\right\rangle = \boxed{\sqrt{\frac{2}{3}}\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle - \frac{1}{\sqrt{6}}\left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}\right\rangle - \frac{1}{\sqrt{6}}\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2}\right\rangle}$$

$$\left|1\frac{1}{2}\frac{-1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|10;\frac{1}{2}\frac{-1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|1-1;\frac{1}{2}\frac{1}{2}\right\rangle$$

$$\boxed{\left|1\frac{1}{2}\frac{-1}{2}\right\rangle = \frac{1}{\sqrt{6}}\left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle + \frac{1}{\sqrt{6}}\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2}\right\rangle}$$

$$\begin{aligned}\left|0\frac{1}{2}\frac{1}{2}\right\rangle &= \left|00;\frac{1}{2}\frac{1}{2}\right\rangle \\ &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle - \left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2}\right\rangle\right)\otimes\left|\frac{1}{2}\frac{1}{2}\right\rangle\end{aligned}$$

$$\left|0\frac{1}{2}\frac{1}{2}\right\rangle = \boxed{\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2}\right\rangle - \left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}\right\rangle\right)}$$

$$\left|0\frac{1}{2}\frac{-1}{2}\right\rangle = \left|00;\frac{1}{2}\frac{-1}{2}\right\rangle$$
$$= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle - \left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2}\right\rangle\right) \otimes \left|\frac{1}{2}\frac{-1}{2}\right\rangle$$

$$\left|0\frac{1}{2}\frac{-1}{2}\right\rangle = \boxed{\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle - \left|\frac{1}{2}\frac{-1}{2};\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{-1}{2}\right\rangle\right)}$$

(c) Identify the symmetry, if any, of the eigenstates of total angular momentum obtained in (b) under the exchange of any two particles.

The symmertic states under exchange of any two particles are:

$$\left|1\tfrac{3}{2}\tfrac{3}{2}\right\rangle \quad \left|1\tfrac{3}{2}\tfrac{1}{2}\right\rangle$$
$$\left|1\tfrac{3}{2}\tfrac{-1}{2}\right\rangle \quad \left|1\tfrac{3}{2}\tfrac{-3}{2}\right\rangle$$

The states the are antisymmetric under exchange of any two particles are:

$$\left|0\tfrac{1}{2}\tfrac{1}{2}\right\rangle \quad \left|0\tfrac{1}{2}\tfrac{-1}{2}\right\rangle$$

PROBLEM 50

Consider two arbitrary vectors whose cartesian coordinates are given by:

$$\boldsymbol{A} = (A_x, A_y, A_z) \text{ and } \boldsymbol{B} = (B_x, B_y, B_z)\,.$$

(a) Obtain the spherical representation of the two vectors $\boldsymbol{A} = (A_+, A_0, A_-)$ and $\boldsymbol{B} = (B_+, B_0, B_-)$ by using the following unit vectors defined in class:

$$\hat{e}_+ \equiv -\frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})\,, \quad \hat{e}_0 \equiv \hat{z}, \quad \hat{e}_- \equiv \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})\,.$$

$$
\begin{aligned}
A_+ &= \vec{A}\cdot\hat{e}_+ \\
&= (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot\frac{-1}{\sqrt{2}}(\hat{x}+i\hat{y}) \\
&= -\frac{1}{\sqrt{2}}(A_x + iA_y) \\
A_0 &= \vec{A}\cdot\hat{e}_0 \\
&= A_z \\
A_- &= \vec{A}\cdot\hat{e}_- \\
&= (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot\frac{1}{\sqrt{2}}(\hat{x}-i\hat{y}) \\
&= \frac{1}{\sqrt{2}}(A_x - iA_y)
\end{aligned}
$$

From $\vec{A}$ it can be shown:

$$
\begin{aligned}
B_+ &= -\frac{1}{\sqrt{2}}(B_x + iB_y) \\
B_0 &= B_z \\
B_- &= \frac{1}{\sqrt{2}}(B_x - iB_y)
\end{aligned}
$$

(b) Build the rank-0 spherical tensor T_{00} from **A** and **B**. Show that $T_{00} \propto \boldsymbol{A}\cdot\boldsymbol{B}$.

$$
\begin{aligned}
\hat{T}_{kq} &= \sum_{\mu\nu} A_{1\mu}B_{1\nu}\langle 1\mu;1\nu|kq\rangle, \qquad (\mu+\nu=q) \\
\hat{T}_{00} &= \sum_{\mu,-\mu} A_{1\mu}B_{1-\mu}\langle 1\mu;1-\mu|00\rangle, \qquad (\mu+\nu=0) \\
&= A_{11}B_{1-1}\langle 11,1-1|00\rangle + A_{10}B_{10}\langle 10,10|00\rangle + A_{1-1}B_{11}\langle 1-1,11|00\rangle
\end{aligned}
$$

$$
\hat{T}_{00} = -\frac{1}{2\sqrt{3}}\left[(A_x+iA_y)(B_x-iB_y)\right] - \frac{1}{\sqrt{3}}A_zB_z - \frac{1}{2\sqrt{3}}\left[(A_x-iA_y)(B_x+iB_y)\right]
$$

$$
\hat{T}_{00} = -\frac{1}{2\sqrt{3}}(A_xB_x \cancel{-iA_xB_y} \cancel{+iA_yB_x} + A_yB_y) - \frac{1}{\sqrt{3}}A_zB_z - \frac{1}{2\sqrt{3}}(A_xB_x \cancel{+iA_xB_y} \cancel{-iA_yB_x} + A_yB_y)
$$

$$\boxed{\hat{T}_{00} = -\frac{1}{\sqrt{3}} \left(A_x B_x + A_y B_y + A_z B_z \right) = -\frac{1}{\sqrt{3}} \left(A \cdot B \right)} \tag{2}$$

(c) Build the rank-0 spherical tensor T_{00} from **A** and **B**. Show that $T_{00} \propto (\boldsymbol{A} \times \boldsymbol{B})_z$.

$$\hat{T}_{10} = \sum_{\mu,-\mu} A_{1\mu} B_{1-\mu} \langle 1\mu; 1-\mu | 00 \rangle \,, \qquad (\mu + \nu = 0)$$

$$\begin{aligned}
\hat{T}_{11} &= A_{10} B_{11} \langle 10; 11 | 11 \rangle + A_{11} B_{10} \langle 11; 10 | 11 \rangle \\
&= \frac{1}{2} A_z \left(B_x + i B_y \right) - \frac{1}{2} \left(A_x + i B_y \right) B_z \\
&= -\frac{1}{2} \left(\vec{A} \times \vec{B} \right)_y - \frac{i}{2} \left(\vec{A} \times \vec{B} \right)_x \\
\hat{T}_{1-1} &= A_{10} B_{1-1} \langle 10; 1-1 | 1-1 \rangle + A_{1-1} B_{10} \langle 1-1; 10 | 1-1 \rangle \\
&= \frac{1}{2} A_z \left(B_x - i B_y \right) + \frac{1}{2} \left(A_x - i A_y \right) B_z \\
&= -\frac{1}{2} \left(\vec{A} \times \vec{B} \right)_y + \frac{i}{2} \left(\vec{A} \times \vec{B} \right)_x
\end{aligned}$$

$$\hat{T}_{10} = A_{10} B_{10} \langle 10; 10 | 10 \rangle + A_{1-1} B_{11} \langle 1-1; 11 | 10 \rangle + A_{11} B_{1-1} \langle 11, 1-1 | 10 \rangle$$

$$\hat{T}_{10} = \frac{1}{2\sqrt{2}} \left(A_x - i A_y \right) \left(B_x + i B_y \right) - \frac{1}{2\sqrt{2}} \left(A_x + i A_y \right) \left(B_x - i B_y \right)$$

$$\hat{T}_{10} = \frac{1}{2\sqrt{2}} (A_x B_x + i A_x B_y - i A_y B_x + A_y B_y - A_x B_x \\ + i A_y B_y - i A_y B_x - A_y B_y)$$

$$\hat{T}_{10} = \frac{i}{\sqrt{2}} \left(A_x B_y - A_y B_x \right)$$

$$\boxed{\hat{T}_{10} = \frac{i}{\sqrt{2}} \left(\vec{A} \times \vec{B} \right)_z} \tag{3}$$

(d) Build the rank-2 spherical tensor T_{2q} from **A** and **B**.

$$\begin{aligned}
\hat{T}_{22} &= A_{10}B_{1-1}\langle 10;1-1|2-1\rangle + A_{1-1}B_{10}\langle 1-1;10|2-1\rangle \\
&= \boxed{\frac{1}{\sqrt{2}}(A_x+iA_y)(B_x+iB_y)} \\
\hat{T}_{21} &= A_{11}B_{10}\langle 11;10|21\rangle + A_{10}B_{11}\langle 10;11|21\rangle \\
&= \boxed{\frac{1}{2}(A_x+iA_y)B_z+\frac{1}{2}A_z(B_x+iB_y)} \\
&= A_{11}B_{1-1}\langle 11;1-1|20\rangle + A_{1-1}B_{11}\langle 1-1;11|20\rangle + A_{10}B_{10}\langle 10;10|20\rangle
\end{aligned}$$

$$\hat{T}_{20} = \frac{1}{2\sqrt{6}}(A_X+iA_y)(B_x+iB_y) + \frac{1}{2\sqrt{6}}(A_x-iA_y)(B_x-iB_y) + \sqrt{\frac{2}{3}}A_zB_z$$

$$\boxed{\hat{T}_{20} = -\frac{1}{\sqrt{6}}(A_xB_x+A_yB_y)+\frac{2}{\sqrt{6}}A_zB_z} \tag{4}$$

$$\begin{aligned}
\hat{T}_{2-1} &= A_{10}B_{1-1}\langle 10;1-1|2-1\rangle + A_{1-1}B_{10}\langle 1-1;10|2-1\rangle \\
&= \boxed{\frac{1}{2}A_z(B_x-iB_y)+\frac{1}{2}(A_x-iA_y)B_z} \\
\hat{T}_{2-2} &= A_{1-1}B_{1-1}\langle 1-1;1-1|2-2\rangle \\
&= \boxed{-\frac{1}{\sqrt{2}}(A_x-iA_y)(B_x-iB_y)}
\end{aligned}$$

PROBLEM 51

Consider the following symmetric, second rank tensor that one can build from the cartesian components of the spatial vector $\boldsymbol{r}$: $\hat{T}_{ij} = \hat{r}_i\hat{r}_j$.

(a) Using the results from Problem 50, construct all the component of the irreducible spherical tensor of rank 2.

$$T_{ij}^{(0)} = \tfrac{1}{3}\delta_{ij}\sum T_{ij}$$

$$\boxed{T_{ij}^{(1)} = \frac{1}{2}(T_{ij}-T_{ji})}$$

$$T_{ij}^{(2)} = \tfrac{1}{2}(T_{ij}+T_{ji})$$

$$
\begin{aligned}
\hat{T}_{00} &= \frac{-1}{\sqrt{3}} \\
\hat{T}_{1q} &= 0 \\
\hat{T}_{22} &= \frac{1}{2}\left(\hat{x}^2 - \hat{y}^2\right) + i\hat{x}\hat{y} \\
\hat{T}_{21} &= \hat{x}\hat{z} + i\hat{y}\hat{z} \\
\hat{T}_{20} &= \frac{-1}{\sqrt{6}}\left(\hat{x}^2 + \hat{y}^2\right) + \frac{2}{\sqrt{6}}\hat{z}^2 \\
\hat{T}_{2-1} &= \hat{x}\hat{z} - i\hat{z}\hat{y} \\
\hat{T}_{2-2} &= \frac{1}{2}\left(\hat{x}^2 - \hat{y}^2\right) - i\hat{x}\hat{y}
\end{aligned}
$$

(b) The *"quadrupole moment"* Q is defined as the following expectation value:

$$
Q \equiv \langle \alpha, j, m = j | \left(2\hat{z}^2 - \hat{x}^2 - \hat{y}^2\right) | \alpha, j, m = j \rangle \,,
$$

where α represents all quantum numbers needed to specify the state of the system other than those associated with the angular momentum. Using the Wigner-Eckart theorem, evaluate the following matrix element:

$$
\langle \alpha, j, m = j | \left(\hat{x}^2 - \hat{y}^2\right) | \alpha, j, m = j \rangle \,,
$$

in terms of Q and appropriate Clebsch-Gordan coefficients.

$$
\begin{aligned}
W &= \left\langle \alpha j m' | T_{22} + T_{2-2} | \alpha j m \right\rangle \\
&= \left\langle j \middle\| T_2 \middle\| j \right\rangle \left(\left\langle jm; 22 \middle| j'm \right\rangle + \left\langle jm; 2-2 \middle| j'm' \right\rangle \right) \\
Q &= \sqrt{6} \left\langle \alpha j j | T_{20} | \alpha j j \right\rangle \\
&= \left\langle j \middle\| T_2 \middle\| j \right\rangle \left\langle jj; 20 | jj \right\rangle \\
W_{mm'} &= \frac{Q}{\sqrt{6}} \left(\frac{\left\langle jm; 22 \middle| jm' \right\rangle + \left\langle jm; 2-2 \middle| jm \right\rangle}{\left\langle jj; 20 \middle| jj \right\rangle} \right)
\end{aligned}
$$