QUANTUM MECHANICS B (PHY5646)
HOMEWORK 16
(February 7, 2017)
Due on Tuesday, February 14, 2017

PROBLEM 46

We have shown in class that the 2×2 Wigner $\mathcal{D}$-matrix for the case of $j = 1/2$ is given by the following expression:

$$\mathcal{D}^{(1/2)}_{m',m}(\alpha,\beta,\gamma) = \exp\left(-\frac{i}{\hbar}(\alpha m' + \gamma m)\right) d^{(1/2)}_{m',m}(\beta),$$

Where

$$d^{(1/2)}_{m',m}(\beta) = \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) & -\sin\left(\frac{\beta}{2}\right) \\ \sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right) \end{pmatrix}.$$

Without looking at the table of Clebsch-Gordan coeffcients, obtain the corresponding 3×3 matrix $d^{(1)}_{m',m}(\beta)$ for the case of $j = 1$. **Hint:** Look back at Problem 39.

$$\begin{aligned}
\mathcal{D}^{1}_{m',m}(\alpha,\beta,\gamma) &= \langle j,m'|\,\hat{U}(\alpha\hat{z})\hat{U}(\beta\hat{y})\hat{U}(\gamma\hat{z})\,|j,m\rangle \\
&= \exp\left(-i(\alpha m' + \gamma m)\right) d^{1}_{m',m}(\beta) \\
d_{m',m} &= \langle 1,m'|\exp\left(-i\beta\hat{L}_y\right)|1,m\rangle \\
e^{-i\beta\hat{L}_y} &= \mathbb{1} - i\sin\theta\hat{L}_y + \left(\cos\theta - 1\right)\hat{L}_y^2 \\
d_{m',m} &= \langle 1,m'|e^{-i\beta m}|1,m\rangle
\end{aligned}$$

$$e^{-i\beta\hat{L}_y} = \mathbb{1} + \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\sin\theta & 0 \\ -\frac{1}{\sqrt{2}}\sin\theta & 0 & \frac{1}{\sqrt{2}}\sin\theta \\ 0 & -\frac{1}{\sqrt{2}}\sin\theta & \end{pmatrix}$$
$$+\begin{pmatrix} \frac{1}{2}\left(\cos\theta - 1\right) & 0 & \frac{1}{2}\left(-\cos\theta + 1\right) \\ 0 & \cos\theta - 1 & 0 \\ \frac{1}{2}\left(-\cos\theta + 1\right) & 0 & \frac{1}{2}\left(\cos\theta - 1\right) \end{pmatrix}$$

$$
\begin{aligned}
e^{-i\beta\hat{L}_y} &= \begin{pmatrix} \frac{1}{2}(\cos\theta+1) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(-\cos\theta+1) \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(-\cos\theta+1) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(\cos\theta+1) \end{pmatrix} \\
&= \begin{pmatrix} d^1_{1,1} & d^1_{1,0} & d^1_{1,-1} \\ d^1_{0,1} & d^1_{0,0} & d^1_{0,-1} \\ d^1_{-1,1} & d^1_{-1,0} & d^1_{-1,1} \end{pmatrix}
\end{aligned}
$$

PROBLEM 47 (Shankar 15.2.2)

By using raising or lowering operators, *but without looking at the table of Clebsch- Gordan coefficients*, derive the Clebsch-Gordan coefficients for the addition of two angular momentum for the following two cases:

(a) $j_1 = 1$ and $j_2 = 1/2$, that is:

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}.$$

$$
\begin{aligned}
|j_1 j_2; jm\rangle &= \sum_{m_1,m_2} |j_1 m_1; j_2 m_2\rangle \boxed{\langle j_1 m_1; j_2 m_2 | j_1 j_2; jm\rangle} \\
\langle \frac{1}{2} m_1; 1 m_2 | \frac{3}{2} m\rangle &= ? \\
J_- \left| \frac{3}{2} \frac{3}{2} \right\rangle &= \sqrt{3}\hbar \left| \frac{3}{2} \frac{1}{2} \right\rangle, \quad \left(\hat{J}_- = \sqrt{(j+m)(j-m+1)}\hbar \right) \\
\left| \frac{3}{2} \frac{1}{2} \right\rangle &= \frac{1}{\sqrt{3}\hbar} J_- \left| \frac{3}{2} \frac{3}{2} \right\rangle \\
&= \frac{1}{\sqrt{3}\hbar} \left(\hat{J}(1)_- + \hat{J}(2)_- \right) \left| \frac{1}{2} \frac{1}{2}, 11 \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\left| \frac{3}{2} \frac{1}{2} \right\rangle = \frac{1}{\sqrt{3}} \sqrt{\left(\frac{1}{2} + \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{2} + 1\right)} \left| \frac{1}{2} \frac{-1}{2}; 11 \right\rangle \\
+ \frac{1}{\sqrt{3}} \sqrt{(1+1)(1-1+1)} \left| \frac{1}{2} \frac{1}{2}; 10 \right\rangle
\end{aligned}
$$

$$
\left| \frac{3}{2} \frac{1}{2} \right\rangle = \boxed{\frac{1}{\sqrt{3}}} \left| \frac{1}{2} \frac{-1}{2}; 11 \right\rangle + \boxed{\sqrt{\frac{2}{3}}} \left| \frac{1}{2} \frac{1}{2}; 10 \right\rangle \tag{1}
$$

$$
\begin{aligned}
\left|\frac{1}{2}\frac{1}{2}\right\rangle &= C_1\left|\frac{1}{2}\frac{-1}{2};11\right\rangle + C_2\left|\frac{1}{2}\frac{1}{2};10\right\rangle \\
\left\langle\frac{3}{2}\frac{1}{2}\middle|\frac{1}{2}\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}C_1 + \sqrt{\frac{2}{3}}C_2 = 0 \\
C_1 &= -\sqrt{2}C_2 \\
(C_1)^2 + (C_2)^2 &= 1 \\
2(C_2)^2 + (C_2)^2 &= 1 \\
C_2 &= \frac{1}{\sqrt{3}}
\end{aligned}
$$

$$
\left|\frac{1}{2}\frac{1}{2}\right\rangle = \boxed{-\sqrt{\frac{2}{3}}}\left|\frac{1}{2}\frac{-1}{2};11\right\rangle + \boxed{\frac{1}{\sqrt{3}}}\left|\frac{1}{2}\frac{1}{2};10\right\rangle \tag{2}
$$

$$
\left\langle\frac{1}{2}\frac{1}{2};11\middle|\frac{3}{2}\frac{1}{2}\right\rangle = \boxed{1} \tag{3}
$$

(b) $j_1 = 1$ and $j_2 = 1$, that is:

$$
1 \otimes 1 = 2 \oplus 1 \oplus 0
$$

Identify the symmetry of the total angular momentum states under the exchange of two identical spin-1 particles.

$$
\begin{aligned}
\langle 1m_1; 1m_2|2m\rangle =?\ \hat{J}_-\,|22\rangle &= 2\hbar\,|21\rangle \\
|2,1\rangle &= \frac{1}{2\hbar}\hat{J}_-\,|22\rangle \\
&= \frac{1}{2\hbar}\left[\hat{J}(1)_- + \hat{J}(2)_-\right]|11;11\rangle
\end{aligned}
$$

$$
|2,1\rangle = \frac{1}{2}\left[\sqrt{(1+1)(1-1+1)}\,|10;11\rangle + \sqrt{(1+1)(1-1+1)}\,|11;10\rangle\right]
$$

$$
|2,1\rangle = \boxed{\frac{\sqrt{2}}{2}}|10;11\rangle + \boxed{\frac{\sqrt{2}}{2}}|11;10\rangle \tag{4}
$$

$$\begin{aligned}
\hat{J}_- |21\rangle &= \sqrt{6}\hbar\, |20\rangle \\
|20\rangle &= \frac{1}{\sqrt{6}\hbar} \hat{J}_- |21\rangle \\
&= \frac{1}{\sqrt{6}\hbar} \left[\hat{J}(1)_- + \hat{J}(2)_-\right] \frac{1}{\sqrt{2}} \left[|11;10\rangle + |10;11\rangle\right]
\end{aligned}$$

$$|20\rangle = \frac{1}{\sqrt{12}} \left(\sqrt{2}\, |10;10\rangle + \sqrt{2}\, |11;1-1\rangle + \sqrt{2}\, |1-1;11\rangle + \sqrt{2}\, |10;10\rangle\right)$$

$$|20\rangle = \boxed{\sqrt{\frac{2}{3}}} |10;10\rangle + \boxed{\frac{1}{\sqrt{6}}} |11;1-1\rangle + \boxed{\frac{1}{\sqrt{6}}} |1-1;11\rangle \tag{5}$$

$$\begin{aligned}
|11\rangle &= C_1\, |10;11\rangle + C_2\, |11;10\rangle && (6) \\
\langle 22;11| &= \frac{\sqrt{2}}{2} C_1 + \frac{\sqrt{2}}{2} C_2 = 0 && (7) \\
C_1 &= -C_2 && (8) \\
(C_1)^2 + (C_2)^2 &= 1 && (9) \\
C_2 &= \frac{\sqrt{2}}{2} && (10)
\end{aligned}$$

$$|11\rangle = \boxed{\frac{-\sqrt{2}}{2}} |10;11\rangle + \boxed{\frac{\sqrt{2}}{2}} |11;10\rangle \tag{11}$$

$$\begin{aligned}
\hat{J}_- |11\rangle &= \sqrt{2}\hbar\, |10\rangle \\
|10\rangle &= \frac{1}{\sqrt{2}\hbar} J_- |11\rangle \\
&= \frac{1}{\sqrt{2}\hbar} \left[\hat{J}(1)_- + \hat{J}(2)_-\right] \frac{1}{\sqrt{2}} \left(|10;11\rangle - |11;10\rangle\right)
\end{aligned}$$

$$|10\rangle = \frac{1}{2} \left(-\sqrt{2}\, |1-1;11\rangle + \sqrt{2}\, |10;10\rangle - \sqrt{2}\, |10;10\rangle + \sqrt{2}\, |11;1-1\rangle\right)$$

$$|10\rangle = \boxed{-\frac{\sqrt{2}}{2}} |1-1;11\rangle + \boxed{\frac{\sqrt{2}}{2}} |11;1-1\rangle \tag{12}$$

$$|00\rangle = C_3 |10;10\rangle + C_4 |11;1-1\rangle + C_5 |1-1;11\rangle \quad (13)$$

$$\langle 20|00\rangle = \sqrt{\frac{2}{3}} C_3 + \frac{1}{\sqrt{6}} C_4 + \frac{1}{\sqrt{6}} C_5 = 0 \quad (14)$$

$$\langle 10|00\rangle = \frac{1}{\sqrt{2}} C_4 + \frac{1}{\sqrt{2}} C_5 = 0 \quad (15)$$

$$C_4 = C_5 = -C_3 \quad (16)$$

$$(C_3)^2 + (C_4)^2 + (C_5)^2 = 1 \quad (17)$$

$$3 (C_4)^2 = 1 \quad (18)$$

$$C_4 = \frac{1}{\sqrt{3}} \quad (19)$$

$$|00\rangle = \boxed{-\frac{1}{\sqrt{3}}} |10;10\rangle + \boxed{\frac{1}{\sqrt{3}}} |11;1-1\rangle + \boxed{\frac{1}{\sqrt{3}}} |1-1;11\rangle \quad (20)$$

PROBLEM 48

Two identical spin-3/2 particles with "frozen" spatial degrees of freedom interact via the following spin-spin Hamiltonian:

$$\hat{H} = \frac{\epsilon_0}{\hbar^2} \hat{S}(1) \cdot \hat{S}(2),$$

where $\epsilon_0 > 0$ is a positive constant that sets the energy scale for the problem.

(a) Obtain the eigenvalues and eigenvector of the Hamiltonian. You should take advantage of the fact that the Hamiltonian is already diagonal in the *total* angular momentum basis $|s, m\rangle$.

$$\begin{aligned}
\left(\hat{S}(1) + \hat{S}(2)\right)^2 &= \hat{S}(1)^2 + 2\hat{S}(1) \cdot \hat{S}(2) + \hat{S}(2)^2 \\
\hat{H} &= \frac{\epsilon_0}{2\hbar^2} \left[\left(\hat{S}(1) + \hat{S}(2)\right)^2 - \left(\hat{S}(1)^2 + \hat{S}(2)^2\right) \right] \\
\langle sm|\hat{H}|sm\rangle &= \frac{\epsilon_0}{2\hbar^2} \langle s'm'| \left[S^2 - \left(\hat{S}(1)^2 + \hat{S}(2)^2\right) \right] |sm\rangle \\
&= \frac{\epsilon_0}{2} \langle s'm'| \left[s(s+1) - (s_1(s_1+1) + s_2(s_2+1)) \right] |sm\rangle \\
&= \frac{\epsilon_0}{2} \langle s'm'| \left[s(s+1) - \left(\frac{15}{4} + \frac{15}{4}\right) \right] |sm\rangle \\
&= \frac{\epsilon_0}{2} \langle s'm'| \left[s(s+1) - \frac{15}{2} \right] |sm\rangle
\end{aligned}$$

	$\|3\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$
$\|3\rangle$	$\frac{9}{4}$	0	0	0
$\|2\rangle$	0	$\frac{-3}{4}$	0	0
$\|1\rangle$	0	0	$\frac{-11}{4}$	0
$\|0\rangle$	0	0	0	$\frac{-15}{4}$

(b) Assume that at time $t = 0$ the two-particle system is prepared in the following anti-symmetric state:

$$|\psi(0)\rangle = \left|\frac{3}{2}, -\frac{3}{2}\right\rangle_A \equiv \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2}, -\frac{3}{2}\right\rangle - \left|\frac{-3}{2}, \frac{3}{2}\right\rangle\right),$$

where the initial state is written as a linear combination of *direct product* states of the form $|m1, m2\rangle \equiv |3/2, m_1\rangle \otimes |3/2, m_2\rangle$. Find the probability that at time $t > 0$ the system will be found in the state

$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle_A \equiv \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle - \left|\frac{-1}{2}, \frac{1}{2}\right\rangle\right).$$

For this part of the problem you may need to use a table of Clebsch-Gordan coefficients.

$$|\psi(0)\rangle = \frac{1}{\sqrt{20}}|30\rangle + \frac{1}{2}|20\rangle + \frac{3}{\sqrt{20}}|10\rangle + \frac{1}{2}|00\rangle - \left(\frac{1}{\sqrt{20}}|30\rangle - \frac{1}{2}|20\rangle + \frac{3}{\sqrt{20}}|10\rangle - \frac{1}{2}|00\rangle\right)$$

$$\begin{aligned}
|\psi(0)\rangle &= \frac{1}{\sqrt{2}}(|20\rangle + |00\rangle) \\
|\psi(t)\rangle &= \sum \frac{1}{\sqrt{2}} e^{\frac{-i\hat{H}t}{\hbar}}(|20\rangle + |00\rangle) \\
&= \boxed{\frac{1}{\sqrt{2}}\left(e^{\frac{3i\epsilon_0 t}{4}}|20\rangle + e^{\frac{15i\epsilon_0 t}{4}}|00\rangle\right)}
\end{aligned}$$

$$|\psi_f\rangle = \frac{3}{\sqrt{20}}|30\rangle + \frac{1}{2}|20\rangle - \frac{1}{\sqrt{20}}|10\rangle - \frac{1}{2}|00\rangle - \left(\frac{3}{\sqrt{20}}|30\rangle - \frac{1}{2}|20\rangle - \frac{1}{\sqrt{20}}|10\rangle + \frac{1}{2}|00\rangle\right)$$

$$
\begin{aligned}
|\psi_f\rangle &= \frac{1}{\sqrt{2}}\left(|20\rangle - |00\rangle\right) \\
\langle\psi_f|\psi(t)\rangle &= \frac{1}{2}\left(\langle 20| - \langle 00|\right)\left(e^{\frac{3i\epsilon_0 t}{4}}\,|20\rangle + e^{\frac{15i\epsilon_0 t}{4}}\,|00\rangle\right) \\
&= \frac{1}{2}e^{\frac{3i\epsilon_0 t}{4}}\left(1 - e^{3i\epsilon_0 t}\right) \\
|\langle\psi_f|\psi(t)\rangle|^2 &= \frac{1}{4}\left(2 - e^{-3i\epsilon_0 t} - e^{3i\epsilon_0 t}\right) \\
&= \frac{1}{4}\left(2 - 2\cos(3\epsilon_0 t)\right)
\end{aligned}
$$

$$
\boxed{P = \frac{1 - \cos(3\epsilon_0 t)}{2}} \tag{21}
$$