QUANTUM MECHANICS A (PHY5645)
HOMEWORK 11
(November 22, 2016)
Due on Thursday, December 1, 2016
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PROBLEM 31 - Shankar 12.3.7 (Two-dimensional Harmonic Oscillator)

Consider a particle of mass μ moving in two spatial dimensions in the presence of an isotropic harmonic oscillator of frequency ω. The Hamiltonian for such a system is given by

$$\hat{H} = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2\mu} + \frac{1}{2}\mu\omega^2\left(\hat{X}^2 + \hat{Y}^2\right).$$

(a) Show that the Hamiltonian commutes with the z-component of the angular momentum, i.e., $\left[\hat{H}, \hat{L}_z\right] = 0$, and use this fact to reduce the eigenvalue problem for $\hat{H}$ to the radial differential equation for $R_{Em}(\rho)$. That is, show that $R_{Em}(\rho)$ satisfies the following differential equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{m^2}{\rho^2} + \frac{2\mu}{\hbar^2}\left(E - \frac{1}{2}\mu\omega^2\rho^2\right)\right] R_{Em}(\rho) = 0.$$

Testing commutativity for the momentum component yields:

$$\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x$$
$$[A, BC] = -[BC, A] = B[A, C] + [A, B]C$$
$$[A, B - C] = [A, B] + [C, A]$$
$$[X_i, X_j] = 0$$
$$[P_i, P_j] = 0$$
$$[X_i, P_j] = i\hbar\delta_{ij}$$

$$\left[\hat{P}_x^2 + \hat{P}_y^2, \hat{L}_z\right] = \left[\hat{P}_x\hat{P}_x, \hat{L}_z\right] + \left[\hat{P}_y\hat{P}_y, \hat{L}_z\right] \tag{1}$$
$$= \hat{P}_x\left[\hat{P}_x, \hat{L}_z\right] + \left[\hat{P}_x, \hat{L}_z\right]\hat{P}_x + \hat{P}_y\left[\hat{P}_y, \hat{L}_z\right] + \left[\hat{P}_x, \hat{L}_z\right]\hat{P}_y \tag{2}$$
$$\tag{3}$$

$$
\begin{aligned}
&= \hat{P}_x \left[\hat{P}_x, \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x\right] + \left[\hat{P}_x, \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x\right] \hat{P}_x \\
&\quad + \hat{P}_y \left[\hat{P}_x, \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x\right] + \left[\hat{P}_y, \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x\right] \hat{P}_y
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&= \hat{P}_x \left(\left[\hat{P}_x, \hat{X}\hat{P}_y\right] + \left[\hat{Y}\hat{P}_x, \hat{P}_x\right]\right) + \left(\left[\hat{P}_x, \hat{X}\hat{P}_y\right] + \left[\hat{Y}\hat{P}_x, \hat{P}_x\right]\right) \hat{P}_x \\
&\quad + \hat{P}_y \left(\left[\hat{P}_y, \hat{X}\hat{P}_y\right] + \left[\hat{Y}\hat{P}_x, \hat{P}_y\right]\right) + \left(\left[\hat{P}_y, \hat{X}\hat{P}_y\right] + \left[\hat{Y}\hat{P}_x, \hat{P}_y\right]\right) \hat{P}_y
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&= \hat{P}_x \left(\hat{X}\left[\hat{P}_x, \hat{P}_y\right] + \left[\hat{P}_x, \hat{X}\right]\hat{P}_y + \hat{P}_x\left[\hat{Y}, \hat{P}_x\right] + \left[\hat{Y}, \hat{P}_x\right]\hat{P}_x\right) \\
&\quad + \left(\hat{X}\left[\hat{P}_x, \hat{P}_y\right] + \left[\hat{P}_x, \hat{X}\right]\hat{P}_y + \hat{P}_x\left[\hat{Y}, \hat{P}_x\right] + \left[\hat{Y}, \hat{P}_x\right]\hat{P}_x\right)\hat{P}_x \\
&\quad + \hat{P}_y \left(\hat{X}\left[\hat{P}_y, \hat{P}_y\right] + \left[\hat{P}_y, \hat{X}\right]\hat{P}_y + \hat{P}_x\left[\hat{Y}, \hat{P}_y\right] + \left[\hat{Y}, \hat{P}_x\right]\hat{P}_y\right) \\
&\quad + \left(\hat{X}\left[\hat{P}_y, \hat{P}_y\right] + \left[\hat{P}_y, \hat{X}\right]\hat{P}_y + \hat{P}_x\left[\hat{Y}, \hat{P}_y\right] + \left[\hat{Y}, \hat{P}_x\right]\hat{P}_y\right)\hat{P}_y
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&\hat{P}_x \left(0 - i\hbar\hat{P}_y + 0 + 0\right) + \left(0 - i\hbar\hat{P}_y + 0 + 0\right)\hat{P}_x \\
&\quad + \hat{P}_y \left(0 + 0 + i\hbar\hat{P}_x + 0\right) + \left(0 + 0 + i\hbar\hat{P}_x + 0\right)\hat{P}_y
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&= -i\hbar\hat{P}_x\hat{P}_y - i\hbar\hat{P}_y\hat{P}_x + i\hbar\hat{P}_x\hat{P}_y + i\hbar\hat{P}_y\hat{P}_x && (8) \\
&= i\hbar\left[\hat{P}_x, \hat{P}_y\right] + i\hbar\left[\hat{P}_x, \hat{P}_y\right] && (9) \\
&= 0 && (10)
\end{aligned}
$$

Testing commutativity for the position component yields:

$$
\left[\hat{X}^2 + \hat{Y}^2, \hat{L}_z\right] = \left[\hat{X}^2, \hat{L}_z\right] + \left[\hat{Y}^2, \hat{L}_z\right] \tag{11}
$$

$$
= \hat{X}\left[\hat{X}, \hat{L}_z\right] + \left[\hat{X}, \hat{L}_z\right]\hat{X} + \hat{Y}\left[\hat{Y}, \hat{L}_z\right] + \left[\hat{Y}, \hat{L}_z\right]\hat{Y} \tag{12}
$$

$$
\begin{aligned}
&\hat{X}\left(\left[\hat{X}, \hat{X}\hat{P}_y\right] + \left[\hat{Y}\hat{P}_x, \hat{X}\right]\right) + \left(\left[\hat{X}, \hat{X}\hat{P}_y\right] + \left[\hat{Y}\hat{P}_x, \hat{X}\right]\right)\hat{X} \\
&\quad + \hat{Y}\left(\left[\hat{Y}, \hat{X}\hat{P}_y\right] + \left[\hat{Y}\hat{P}_x, \hat{Y}\right]\right) + \left(\left[\hat{Y}, \hat{X}\hat{P}_y\right] + \left[\hat{Y}\hat{P}_x, \hat{Y}\right]\right)\hat{Y}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&= \hat{X}\left(\hat{X}\left[\hat{P}_y,\hat{X}\right]+\left[\hat{P}_y,\hat{X}\right]\hat{X}+\hat{Y}\left[\hat{X},\hat{P}_x\right]+\left[\hat{X},\hat{Y}\right]\hat{P}_x\right)\\
&\quad+\left(\hat{X}\left[\hat{P}_y,\hat{X}\right]+\left[\hat{P}_y,\hat{X}\right]\hat{X}+\hat{Y}\left[\hat{X},\hat{P}_x\right]+\left[\hat{X},\hat{Y}\right]\hat{P}_x\right)\hat{X}\\
&\quad+\hat{Y}\left(\hat{Y}\left[\hat{P}_y,\hat{X}\right]+\left[\hat{P}_y,\hat{Y}\right]\hat{X}+\hat{Y}\left[\hat{Y},\hat{P}_x\right]+\left[\hat{Y},\hat{Y}\right]\hat{P}_x\right)\\
&\quad+\left(\hat{Y}\left[\hat{P}_y,\hat{X}\right]+\left[\hat{P}_y,\hat{Y}\right]\hat{X}+\hat{Y}\left[\hat{Y},\hat{P}_x\right]+\left[\hat{Y},\hat{Y}\right]\hat{P}_x\right)\hat{Y}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&\hat{X}\left(0+0+i\hbar\hat{Y}+0\right)+\left(0+0+i\hbar\hat{Y}+0\right)\hat{X}\\
&\quad+\hat{Y}\left(0-i\hbar\hat{X}+0+0\right)+\left(0-i\hbar\hat{X}+0+0\right)\hat{Y}
\end{aligned}
\tag{15}
$$

$$= i\hbar\hat{X}\hat{Y}+i\hbar\hat{Y}\hat{X}-i\hbar\hat{Y}\hat{X}-i\hbar\hat{X}\hat{Y} \tag{16}$$

$$= i\hbar\left[\hat{X},\hat{Y}\right]+i\hbar\left[\hat{Y},\hat{X}\right] \tag{17}$$

$$= 0 \tag{18}$$

$$\text{Therefore, } \left[\hat{H},\hat{L}_z\right] = 0 \tag{19}$$

Changing the Hamiltonian into polar coordinates:

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2+\frac{1}{2}\mu\omega^2\rho^2 \tag{20}$$

$$
\begin{gathered}
\nabla^2 = \tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\\
\rho^2 = x^2+y^2\\
x = \rho\cos\phi\\
y = \rho\sin\phi\\
\tfrac{\partial x}{\partial\rho} = \cos\phi\\
\tfrac{\partial x}{\partial\phi} = -\rho\sin\phi\\
\tfrac{\partial y}{\partial\rho} = \sin\phi\\
\tfrac{\partial y}{\partial\phi} = \rho\cos\phi
\end{gathered}
$$

$$\frac{\mathrm{d}}{\mathrm{d}\rho} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \rho} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}\phi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \phi} = -\rho\sin\phi\frac{\partial}{\partial x} + \rho\cos\phi\frac{\partial}{\partial y} \tag{22}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} = \frac{\partial}{\partial \rho}\left(\cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y}\right) \tag{23}$$

$$= \cos\phi\left(\frac{\partial^2}{\partial x^2}\frac{\partial x}{\partial \rho} + \frac{\partial^2}{\partial x\partial y}\frac{\partial y}{\partial \rho}\right) + \sin\phi\left(\frac{\partial^2}{\partial y^2}\frac{\partial y}{\partial \rho} + \frac{\partial^2}{\partial x\partial y}\frac{\partial x}{\partial \rho}\right) \tag{24}$$

$$= \cos^2\phi\frac{\partial^2}{\partial x^2} + 2\sin\phi\cos\phi\frac{\partial^2}{\partial x\partial y} + \sin^2\phi\frac{\partial^2}{\partial y^2} \tag{25}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} = \rho\frac{\partial}{\partial \rho}\left(-\sin\phi\frac{\partial}{\partial x} + \cos\phi\frac{\partial}{\partial y}\right) \tag{26}$$

$$\tag{27}$$

$$\begin{aligned} &= -\rho\cos\phi - \rho\sin\phi\frac{\partial}{\partial y} - \rho\sin\phi\left(\frac{\partial^2}{\partial x^2}\frac{\partial x}{\partial \phi} + \frac{\partial^2}{\partial x\partial y}\frac{\partial y}{\partial \phi}\right) \\ &\quad + \rho\cos\phi\left(\frac{\partial^2}{\partial y^2}\frac{\partial y}{\partial \phi} + \frac{\partial^2}{\partial x\partial y}\frac{\partial x}{\partial \phi}\right) \end{aligned} \tag{28}$$

$$= -\rho\frac{\partial}{\partial \rho} + \rho^2\sin^2\phi\frac{\partial^2}{\partial x^2} - 2\rho\sin\phi\cos\phi\frac{\partial^2}{\partial x\partial y} + \rho^2\cos^2\phi\frac{\partial^2}{\partial y^2} \tag{29}$$

$$\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} = -\frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{30}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \tag{31}$$

Changing the wavefunction into a function of polar variables:

$$\begin{aligned}
\hat{L}_z &= -i\hbar\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) && (32)\\
&= -i\hbar\left(-\rho\sin\phi\frac{\partial}{\partial x} + \rho\cos\phi\frac{\partial}{\partial y}\right) && (33)\\
&= -i\hbar\frac{\partial}{\partial\phi} && (34)\\
\hat{L}_z\left|\psi\right\rangle &= m\hbar\left|\psi\right\rangle && (35)\\
-i\hbar\frac{\partial}{\partial\phi}\left|\psi(\rho,\phi)\right\rangle &= m\hbar\left|\psi(\rho,\phi)\right\rangle && (36)\\
\psi(\rho,\phi) &= R_{Em}(\rho)e^{im\phi} && (37)\\
\frac{\partial^2}{\partial\phi^2} &= -m^2 && (38)
\end{aligned}$$

Now plugging in the Laplacian and see setting the hamiltonian equal to energy we obtain

$$\begin{aligned}
\hat{H} &= \left[\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right) - \frac{2\mu}{\hbar^2}\left(E - \frac{1}{2}\mu\omega^2\rho^2\right)\right]R_{Em}(\rho) = 0 && (39)\\
& && (40)
\end{aligned}$$

$$\boxed{= \left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{m^2}{\rho^2} + \frac{2\mu}{\hbar^2}\left(E - \frac{1}{2}\mu\omega^2\rho^2\right)\right]R_{Em}(\rho) = 0} \qquad (41)$$

(b) As in the case of the one-dimensional harmonic oscillator, introduce the following two dimensionless variables

$$\epsilon = \frac{E}{\hbar\omega} \qquad \text{and} \qquad y = \frac{\rho}{b} \equiv \sqrt{\frac{\mu\omega}{\hbar}}\rho$$

to rewrite the differential equation as

$$R''_{Em}(y) + \frac{1}{y}R'_{Em}(y) + \left(2\epsilon - \frac{m^2}{y^2} - y^2\right)R_{Em}(y) = 0$$

$$\begin{aligned}
\frac{1}{b^2}R''_{Em} + \frac{1}{b}\frac{1}{\rho}R'_{Em} - \frac{m^2}{\rho^2}R_{Em} + \frac{2\mu}{\hbar^2}E - \frac{\mu^2\omega^2}{\hbar^2}\rho^2 &= 0 && (42)\\
\frac{1}{b^2}R''_{Em} + \frac{1}{y}\frac{1}{b^2}R'_{Em} + \left(\frac{2\epsilon}{b^2} - \frac{1}{y^2}\frac{m^2}{b^2} - \frac{y^2}{b^2}\right)R_{Em} &= 0 && (43)
\end{aligned}$$

Multiplying by b^2 on both sides yields

$$\boxed{R''_{Em}(y) + \frac{1}{y}R'_{Em}(y) + \left(2\epsilon - \frac{m^2}{y^2} - y^2\right) R_{Em}(y) = 0} \tag{44}$$

(c) By examining the limits as $y \to 0$ and as $y \to \infty$ one can motivate rewriting R_{Em} as

$$R_{Em}(y) = y^{|m|}e^{-y^2/2}U(y).$$

Show that $U(y)$ satisfies the following differential equation:

$$U''(y) + \left(\frac{2|m|+1}{y} - 2y\right) U'(y) + (2\epsilon - 2|m| - 2)\, U(y) = 0$$

$$\begin{aligned} R'_{Em} &= \frac{\partial}{\partial y}\left(y^{|m|}e^{-y^2/2}\right) U(y) + \left(y^{|m|}e^{-y^2/2}\right) U'(y) && (45) \\ &= \left(|m|y^{|m|-1} - y^{|m|+1}e^{-y^2/2}\right) U(y) + y^{|m|}e^{-y^2/2}U'(y) && (46) \\ &= y^{|m|}e^{-y^2/2}\left[\left(|m|y^{-1} - y\right) U(y) + U'(y)\right] && (47) \end{aligned}$$

$$\begin{aligned} R''_{Em} &= |m|\frac{\partial}{\partial y}\left(y^{|m|-1}e^{-y^2/2}\right) U(y) + \left(|m|y^{|m|-1}e^{-y^2/2}\right) U'(y) \\ &- \frac{\partial}{\partial y}\left(y^{|m|+1}e^{-y^2/2}\right) U(y) - \left(y^{|m|+1}e^{y^2/2}\right) U'(y) + \frac{\partial}{\partial y}\left(y^{|m|}e^{-y^2/2}\right) U'(y) \\ &+ \left(y^{|m|}e^{-y^2/2}\right) U''(y) \end{aligned} \tag{48}$$

$$\begin{aligned} &= y^{|m|}e^{-y^2/2}[\left(|m|(|m|-1)\, y^{-2} - 2|m| + y^2 - 1\right) U(y) + \left(2|m|y^{-1} - 2y\right) U'(y) \\ &+ U''(y)] \end{aligned} \tag{49}$$

Plugging in for R''_{Em}, R'_{Em} and R_{Em} and factoring out $y^{|m|}e^{-y^2/2}$ gives

$$\boxed{U''(y) + \left(\frac{2|m|+1}{y} - 2y\right) U'(y) + (2\epsilon - 2|m| - 2)\, U(y) = 0} \tag{50}$$

(d) Finally, by changing variables one last time to $x = y^2$, show that $U(x)$ satisfies the following differential equation

$$xU''(x) + (\beta - x)\, U'(x) - \alpha U(x) = 0$$

$$U(y)' = 2yU'(x)$$
$$U''(y) = 4y^2U''(x) + 2U'(x)$$

$$4y^2U''(x) + 2U'(x) + \frac{2y}{y}\left(2|m|+1 - 2y^2\right)U'(x) + 2\left(\epsilon - |m|-1\right)U(x) \quad (51)$$
$$4xU''(x) + 4\left(|m|+1 - x\right)U'(x) - \alpha U(x) = 0 \quad (52)$$
$$\beta = |m|+1, \qquad \alpha = \frac{-\epsilon + |m|+1}{2} \quad (53)$$
$$\boxed{xU''(x) + (\beta - x)U'(x) - \alpha U(x) = 0} \quad (54)$$

PROBLEM 32 - Shankar 12.3.7 (Two-dimensional Harmonic Oscillator)

The differential equation obtained in part (d) of Problem 34 has as a solution the *degenerate hypergeometric function (also called Kummer's confluent hypergeometric function)* $F(\alpha, \beta, x)$ that is defined in terms of an infinite power series in x as follows:

$$F(\alpha, \beta, x) = 1 + \frac{\alpha}{\beta}x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{x^3}{3!} + ... \quad (1)$$

In general, the degenerate hypergeometric function $F(\alpha, \beta, x)$ increases for $x \gg 1$ as $F(\alpha, \beta, x) \propto e^x$, so the radial component of the wave function R_{Em} will diverge at large distances as $R_{Em}(y)/e^{y^2/2}$. However, if the series in Eq. (1) can be made to terminate, then $F(\alpha, \beta, x)$ reduces to a polynomial and $R_{Em}(y)$ will *decrease* at large distances, as required.

(a) Find that the values of α that are required to reduce $F(\alpha, \beta, x)$ to a polynomial of degree $k = 0, 1, 2, ...$ and show that the corresponding quantized energies are given by

$$\epsilon_{km} = (2k + |m|+1) \equiv (n+1).$$

$\alpha = 0$	1
$\alpha = -1$	$1 - \frac{x}{\beta}$
$\alpha = -2$	$1 - \frac{2x}{\beta} + \frac{x^2}{\beta(\beta+1)}$

$$\alpha = -k \quad (55)$$
$$\alpha = \frac{-\epsilon + |m|+1}{2} \quad (56)$$
$$\epsilon_{km} = \boxed{(2k + |m|+1) \equiv (n+1)} \quad (57)$$

(b) For a given value of n, what are the allowed values of $|m|$? Given this information, show that for a fixed value of n the degeneracy is $n+1$. Compare this value of the degeneracy to the answer that you have obtained in Cartesian coordinates where the energies are given by

$$\epsilon_{nx}\epsilon_{ny} = (n_x + n_y + 1) \equiv (n + 1)$$

From the equation $n = 2k + |m|$, it can be seen that n can be any natural number. for $|m|= 0$, $k = n/2$ and for $|m|= 1$, $k = (n-1)/2$. This causes a split for even and odd values of n.

$$|m| \equiv n - 2k \tag{58}$$

$k = 0$	$\|m\|= n$
$k = 1$	$\|m\|= n - 2$
$k = 2$	$\|m\|= n - 4$
$\vdots$	$\vdots$
$k = n$	$\|m\|= n$

For fixed even values of n the allowed values of $|m|$ are

$$|m| \in \{n, n-2, n-4, n-6, \ldots, 0\} \tag{59}$$

For fixed odd values of n, the allowed values of $|m|$ are:

$$|m| \in \{n, n-2, n-4, n-6, \ldots, 1\} \tag{60}$$

There are a total of $\frac{n}{2} + 1$ possibilities for $|m|$ with the lowest value of $|m|$ ending at either 1 or 0 for odd or even values of n, respectively. Since m can be either positive or negative there are $\frac{n}{2}$ more possibilities which gives a total of $n + 1$ possible values. This is the degeneracy of the n-th energy level in both the odd and even case.

$$(n_x + n_y) = n \tag{61}$$

We have two decoupled oscillators, therefore $n_x \in \{0, \ldots, n\}$ and $n_2 = n - n_x$. This degeneracy is equal to $n + 1$

(c) Write down all the normalized eigenfunctions (in polar coordinates) corresponding to $n = 0$ and $n = 1$.

For $n = 0$, $n = 2k + |m|$, therefore $m = k = 0$.

$$
\begin{aligned}
\epsilon &= 1, \quad \alpha = 0, \quad , \beta = 1, F(0, 1, x) = 1 && (62)\\
\psi_{\epsilon,k} &= R_{Em}\Phi(\phi) && (63)\\
&= Ay^{|m|}e^{-y^2/2}F(\alpha, \beta, x)e^{-im\phi} && (64)\\
&= Ae^{-y^2/2}F(0, 1, x) && (65)\\
\langle\psi|\psi\rangle &= |A|^2\int_0^{2\pi}\int_0^{\infty}\rho e^{\rho^2/b^2}\mathrm{d}\rho\mathrm{d}\phi && (66)\\
&= -\frac{2\pi|A|^2b^2}{2}\int_0^{\infty}e^u\mathrm{d}u && (67)\\
&= -\pi|A|^2b^2e^{\rho^2/b^2}\Big|_0^{\infty} && (68)\\
1 &= \pi|A|^2b^2 && (69)\\
|A|^2 &= \frac{1}{\pi b^2} && (70)\\
A &= \sqrt{\frac{\mu\omega}{\pi\hbar}} && (71)\\
\psi_{0,0} &= \boxed{\sqrt{\frac{\mu\omega}{\pi\hbar}}e^{-\mu\omega\rho^2/(2\hbar)}} && (72)
\end{aligned}
$$

For $n = 1$, $m \pm 1$, $k = 0$:

$$\begin{aligned}
\epsilon &= 2, \quad \alpha = 0, \quad F(2,0,x) = 1 && (73)\\
\psi_{\epsilon,k} &= R_{Em}\Phi(\phi) && (74)\\
&= Ay^{|m|}e^{-y^2/2}F(\alpha,\beta,x)e^{-im\phi} && (75)\\
&= Aye^{-y^2/2}F(0,2,x)e^{\pm i\phi} && (76)\\
\langle\psi|\psi\rangle &= \frac{|A|^2}{b^2}\int_0^{2\pi}\int_0^{\infty}\rho^3 e^{-\rho^2/b^2}\mathrm{d}\rho\mathrm{d}\phi && (77)\\
&= \frac{2\pi|A|^2}{b^2}\int_0^{\infty}\rho^3 e^{-\rho^2/b^2}\mathrm{d}\rho\mathrm{d}\phi && (78)\\
&= \frac{2\pi|A|^2}{b^2}\left(\frac{b^4}{2}\right) && (79)\\
&= \pi|A|^2 b^2 && (80)\\
A &= \sqrt{\frac{\mu\omega}{\pi\hbar}} && (81)\\
\psi_{1,\pm1} &= \boxed{\frac{\mu\omega}{\sqrt{\pi}\hbar}\rho e^{-\rho^2/b^2}e^{\pm im\phi}} && (82)
\end{aligned}$$

(d) Argue that the $n = 0$ wave function must equal the corresponding one in Cartesian coordinates and that the $n = 1$ solutions are linear combinations of the Cartesian counterparts.

The $n = 0$ wavefunction has degeneracy 1, therefore it must equal to the one in cartesian coordinates:

$$\psi_{0,0} = \sqrt{\frac{\mu\omega}{\pi\hbar}}e^{-\mu\omega\left(x^2+y^2\right)(2\hbar)} \tag{83}$$

Solutions for $n = 1$ can be seen to be a linear combination of the coordinate solutions:

$$\begin{aligned}
\psi_{1,\pm1} &= A(\rho\cos\phi \pm i\rho\sin\phi)e^{-\rho^2/b^2} && (84)\\
&= (x \pm iy)\, e^{-\left(x^2+y^2\right)/b^2} && (85)\\
& && (86)
\end{aligned}$$

PROBLEM 33 - Shankar 12.3.8 (Landau Levels)

Consider a particle of mass μ and charge q in the presence of a vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{B}{2}(-y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}}),$$

where B is the constant magnitude of the magnetic field.

(a) Show that the magnetic field is given by $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. Incidentally, show that both $\boldsymbol{A}'(\boldsymbol{r}) = -yB\hat{\boldsymbol{x}}$ and $\boldsymbol{A}''(\boldsymbol{r}) = xB\hat{\boldsymbol{y}}$ also generate the same exact magnetic field. Do you understand the significance of these equivalent vector potentials?

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{B}{2}y & \frac{B}{2}x & 0 \end{vmatrix} && (87)\\
&= \cancel{\left(-\frac{B}{2}\frac{\partial x}{\partial z}\right)}\hat{x} + \cancel{\left(-\frac{B}{2}\frac{\partial y}{\partial z}\right)}\hat{y} + \left(\frac{B}{2}\frac{\partial x}{\partial x} + \frac{B}{2}\frac{\partial y}{\partial y}\right)\hat{z} && (88)\\
&= B\hat{z} && (89)\\
\vec{\nabla} \times \vec{A}' &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yB & 0 & 0 \end{vmatrix} && (90)\\
&= \cancel{\left(B\frac{\partial y}{\partial z}\right)}\hat{y} + \left(B\frac{\partial y}{\partial y}\right)\hat{z} && (91)\\
&= B\hat{z} && (92)\\
\vec{\nabla} \times \vec{A}'' &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xB & 0 \end{vmatrix} && (93)\\
&= \cancel{\left(B\frac{\partial x}{\partial z}\right)}\hat{x} + \left(B\frac{\partial x}{\partial x}\right)\hat{z} && (94)\\
&= B\hat{z} && (95)
\end{aligned}$$

The first and second derivative of the vector potential is always perpendicular to the plane.

(b) Show that *classically* a particle in this potential will move in a "cyclotron" orbit, i.e., in a circle at the cyclotron frequency $\omega_0 = qB/\mu c$.

$$v = r\omega_0 \tag{96}$$

$$\frac{\mu v^2}{r} = \frac{qvB}{c} \tag{97}$$

$$\mu r\omega_0^2 = \frac{qr\omega_0 B}{c} \tag{98}$$

$$\omega_0 = \boxed{\frac{qB}{\mu c}} \tag{99}$$

(c) Consider the Hamiltonian for the corresponding quantum problem:

$$\hat{H} = \frac{\left(\hat{P}_x + qB\hat{Y}/2c\right)^2}{2\mu} + \frac{\left(\hat{P}_y - qB\hat{X}/2c\right)^2}{2\mu}$$

Show that the operators $\hat{Q} = \left(c\hat{P}_x + qB\hat{Y}/2\right)/qB$ and $\hat{P} = \left(\hat{P}_y - qB\hat{X}/2c\right)$ are canonical, i.e., they satisfy the fundamental commutation relation: $\left[\hat{Q}, \hat{P}\right] = i\hbar$.

$$\left(\frac{\hat{P}_x}{\mu\omega_0} + \frac{\hat{Y}}{2}\right)\left(\hat{P}_y - \frac{\mu\omega_0\hat{X}}{2}\right) - \left(\hat{P}_y - \frac{\mu\omega_0\hat{X}}{2}\right)\left(\frac{\hat{P}_x}{\mu\omega_0} + \frac{\hat{Y}}{2}\right) \tag{100}$$

$$= \frac{\hat{P}_x\hat{P}_y}{\mu\omega_0} - \frac{\hat{P}_x\hat{X}}{2} + \frac{\hat{Y}\hat{P}_y}{2} - \frac{\mu\omega_0}{2}\hat{Y}\hat{X} - \left(\frac{\hat{P}_y\hat{P}_x}{\mu\omega_0} - \frac{\hat{X}\hat{P}_x}{2} + \frac{\hat{P}_y\hat{Y}}{2} - \frac{\mu\omega_0}{2}\hat{X}\hat{Y}\right) \tag{101}$$

$$= \frac{1}{\mu\omega_0}\cancel{\left[\hat{P}_x, \hat{P}_y\right]} + \frac{1}{2}\left[\hat{X}, \hat{P}_x\right] + \frac{1}{2}\left[\hat{Y}, \hat{P}_y\right] + \frac{\mu\omega_0}{2}\cancel{\left[\hat{X}, \hat{Y}\right]} \tag{102}$$

$$\left[\hat{Q}, \hat{P}\right] = \frac{1}{2}i\hbar + \frac{1}{2}i\hbar \tag{103}$$

$$= i\hbar \tag{104}$$

(d) Write the Hamiltonian in terms of the transformed operators $\hat{P}$ and $\hat{Q}$ and show that the allowed energy levels are given by $E_n = (n + 1/2)\hbar\omega_0$.

$$\hat{H} = \frac{q^2B^2}{c^2}\frac{\hat{Q}^2}{2\mu} + \frac{\hat{P}^2}{2\mu} \tag{105}$$

$$= \boxed{\frac{\hat{P}^2}{2\mu} + \frac{1}{2}\mu\omega_0^2\hat{Q}^2} \tag{106}$$

This is the Hamiltonian of a harmonic oscillator and it is known that the energies are given as $E_n = (n + 1/2)\,\hbar\omega_0$

(e) Expand the Hamiltonian in terms of the original operators $\left(\hat{P}_x, \hat{P}_y, \hat{X}, \hat{Y}\right)$ and show that it can be written as follows:

$$\hat{H} = \hat{H}_0\left(\frac{\omega_0}{2}, \mu\right) - \frac{1}{2}\omega_0 \hat{L}_z,$$

where H_0 is the Hamiltonian of an isotropic two-dimensional harmonic oscillator of mass μ and frequency $\omega_0/2$ and $\hat{L}_z$ is the z-component of the angular-momentum operator. Argue that the same basis that diagonalizes $\hat{H}_0$ will also diagonalize $\hat{H}$.

$$\begin{aligned}
\hat{H} &= \frac{\left(\hat{P}_x^2 + \frac{qB}{c}\hat{Y}\hat{P}_x + \left(\frac{qB}{2c}\right)^2 \hat{Y}^2\right)}{2\mu} + \frac{\left(\hat{P}_y^2 - \frac{qB}{c}\hat{X}\hat{P}_y + \left(\frac{qB}{2c}\right)^2 \hat{X}^2\right)}{2\mu} && (107)\\
&= \frac{\hat{P}_x^2 + \hat{P}_y^2}{2\mu} + \frac{1}{4}\mu\frac{\omega_0^2}{2}\left(\hat{X}^2 + \hat{Y}^2\right) + \frac{\omega_0}{2}\left(\hat{Y}\hat{P}_x - \hat{X}\hat{P}_y\right) && (108)\\
&= \left\{\frac{\hat{P}_x^2 + \hat{P}_y^2}{2\mu} + \frac{1}{2}\mu\left(\frac{\omega_0}{2}\right)^2\left(\hat{X}^2 + \hat{Y}^2\right)\right\} - \frac{1}{2}\omega_0\hat{L}_z && (109)
\end{aligned}$$

It can be seen that the quantity is braces $\{\hat{H}_0\}$ is the a Hamiltonian for the harmonic oscillator similar to the Hamiltonian in prob. 31. By inspection, H_0 then must commute with L_z and L_z must also commute with the last term in the Hamiltonian $\hat{H}$, Therefore, the same basis that diagonalized $\hat{H}_0$ will also diagonalize $\hat{H}$

$$\begin{aligned}
\left[\hat{H}_0, \hat{L}_z\right] &= 0 && (110)\\
\left[\hat{L}_z, \hat{L}_z\right] &= 0 && (111)\\
\therefore \left[\hat{H}, \hat{L}_z\right] &= 0 && (112)\\
& && (113)
\end{aligned}$$

$$\therefore U^\dagger \hat{H}_0 U = \Lambda_0 \text{ and } U^\dagger \hat{H} U = \Lambda \qquad (114)$$

(f) By thinking in terms of the basis found in (e), show that allowed energy levels are given by $E_{km} = \frac{1}{2}(2k + |m| + m + 1)\hbar\omega_0$, where k is any integer and

m is the z-component of the angular momentum. Convince yourself that you get the same energy levels from this formula as from the one obtained in part (d).

From prob. 31 our energy is equal to a value $E = C\,(2k + |m|{+}1)$ from $\hat{H}_0$ of the hamiltonian. $C = 1/2$ comes from the value of ω_0 being halved. we add m to E from the L_z component in the Hamiltonian $\hat{H}$. For the energy levels, the equation becomes $E_{km} = \frac{1}{2}\,(2k + |m|{+}m + 1)\,\hbar\omega_0 \equiv (n+1)\,\hbar\omega$, where $2k + |m|{+}m \equiv n$.