QUANTUM MECHANICS A (PHY5645)
HOMEWORK 10
(November 8, 2016)
Due on Tuesday, November 22, 2016

PROBLEM 28 - Shankar Exercise 7.4.8

Consider the three components of the angular momentum in classical mechanics:

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} \text{ or } l_j = \epsilon_{jkn} x_k p_n = \begin{cases} yp_z - zp_y & \text{if } j = 1 \\ zp_x - xp_z & \text{if } j = 2 \\ xp_y - yp_x & \text{if } j = 3 \end{cases}$$

(a) Promote the classical angular momentum variables l_j to quantum mechanical Hermitian operators $\hat{L}_j$ and show that there are no *ordering ambiguities.*

$$\hat{L}_j = \epsilon_{jkn} \hat{X}_k \hat{P}_n = \begin{cases} \hat{Y}\hat{P}_z - \hat{Z}P_y \\ \hat{Z}\hat{P}_x - \hat{X}\hat{P}_z \\ \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x \end{cases} \tag{1}$$

$$[\hat{X}, \hat{P}_x] = [\hat{Z}, \hat{P}_z] = [\hat{X}, \hat{P}_y] = i\hbar \tag{2}$$

Each product involves a coordinate and momentum that commute (non-zero).

(b) Verify that the classical angular momentum variables satisfy the following relations between Poisson brackets:

$$\{l_j, l_k\} = \epsilon_{jkn} l_n$$

$$\begin{aligned} \{l_j, l_k\} &= \frac{\partial l_j}{\partial x_n}\frac{\partial l_k}{\partial p_n} - \frac{\partial l_j}{\partial p_n}\frac{\partial l_k}{\partial x_n} && (3) \\ l_j &= \epsilon_{jkn} x_k p_n && (4) \\ l_k &= \epsilon_{knj} x_n p_j && (5) \\ l_n &= \epsilon_{njk} x_j p_k && (6) \end{aligned}$$

$$\frac{\partial l_j}{\partial x_n} = -p_k, \quad \frac{\partial l_k}{\partial p_n} = -x_j, \quad \frac{\partial l_j}{\partial p_n} = x_k, \quad , \frac{\partial l_k}{\partial x_n} = p_j \tag{7}$$

$$\{l_j, l_k\} = x_j p_k - x_k p_j = l_n \tag{8}$$

(c) Using the fundamental commutation relations, i.e., $[\hat{X}_j, \hat{P}_k] = i\hbar\delta_{jk}$, verify that the quantum angular momentum operators satisfy the following commutation relations:

$$[\hat{L}_j, \hat{L}_k] = i\hbar\epsilon_{jkn}\hat{L}_n.$$

$$\begin{aligned}
[\hat{L}_j, \hat{L}_k] &= [\hat{X}_k\hat{P}_n - \hat{X}_n\hat{P}_k, \hat{X}_n\hat{P}_j - \hat{X}_j\hat{P}_n] && (9)\\
\left[\hat{A} - \hat{B}, \hat{C} - \hat{D}\right] &= (\hat{A} - \hat{B})(\hat{C} - \hat{D}) - (\hat{C} - \hat{D})(\hat{A} - \hat{B}) && (10)\\
&= [A, C] + [D, A] + [C, B] + [B, D] && (11)\\
& && (12)
\end{aligned}$$

$$\left[\hat{X}_k\hat{P}_n, \hat{X}_n\hat{P}_j\right] + \left[\hat{X}_j\hat{P}_n, \hat{X}_k\hat{P}_n\right] + \left[\hat{X}_n\hat{P}_j, \hat{X}_n\hat{P}_k\right] + \left[\hat{X}_n\hat{P}_k, \hat{X}_j\hat{P}_n\right] \tag{13}$$

$$\left[\hat{X}_j\hat{P}_n, \hat{X}_k\hat{P}_n\right] = \hat{X}_k\left[\hat{X}_j\hat{P}_n, \hat{P}_n\right] + \left[\hat{X}_j\hat{P}_n, \hat{P}_n\right]\hat{P}_n \tag{14}$$

$$= \hat{X}_k\left[\hat{X}_j\left[\cancel{\hat{P}_n, \hat{P}_n}\right] + \left[\cancel{\hat{X}_j, \hat{P}_n}\right]\hat{P}_n\right] + \left[\hat{X}_j\left[\cancel{\hat{P}_n, \hat{P}_n}\right] + \left[\cancel{\hat{X}_j, \hat{P}_n}\right]\hat{P}_n\right]\hat{P}_n \tag{15}$$

$$= 0 \tag{16}$$

$$\left[\hat{X}_n\hat{P}_j, \hat{X}_n\hat{P}_k\right] = \hat{X}_n\left[\hat{X}_n\hat{P}_j, \hat{P}_k\right] + \left[\hat{X}_n\hat{P}_j, \hat{P}_k\right]\hat{P}_j \tag{17}$$

$$= \hat{X}_n\left[\hat{X}_n\left[\cancel{\hat{P}_j, \hat{P}_k}\right] + \left[\cancel{\hat{X}_n, \hat{P}_k}\right]\hat{P}_k\right] + \left[\hat{X}_n\left[\cancel{\hat{P}_j, \hat{P}_k}\right] + \left[\cancel{\hat{X}_n, \hat{P}_k}\right]\hat{P}_j\right]\hat{P}_j \tag{18}$$

$$= 0 \tag{19}$$

$$\begin{aligned}
\left[\hat{X}_k\hat{P}_n, \hat{X}_n\hat{P}_j\right] &= \hat{X}_n\left[\hat{X}_k\hat{P}_n, \hat{P}_j\right] + \left[\hat{X}_k\hat{P}_n, \hat{X}_n\right]\hat{P}_j && (20)\\
\left[\hat{X}_k\hat{P}_n, \hat{P}_j\right] &= \hat{X}_k\left[\cancel{\hat{P}_n, \hat{P}_j}\right] + \left[\cancel{\hat{X}_n, \hat{P}_j}\right]\hat{P}_n && (21)\\
\left[\hat{X}_k\hat{P}_n, \hat{X}_n\right] &= \hat{X}_k\left[\hat{P}_n, \hat{X}_n\right] + [\cancel{\hat{X}_k, \hat{X}_n}]\hat{P}_n && (22)\\
\left[\hat{X}_k\hat{P}_n, \hat{X}_n\hat{P}_j\right] &= \hat{X}_k\left[\hat{P}_n, \hat{X}_n\right]\hat{P}_j && (23)
\end{aligned}$$

$$\begin{aligned}
\left[\hat{X}_n\hat{P}_k, \hat{X}_j\hat{P}_n\right] &= \hat{X}_j\left[\hat{X}_n\hat{P}_k, \hat{P}_n\right] + \left[\hat{X}_n\hat{P}_k, \hat{X}_j\right]\hat{P}_n && (24)\\
\left[\hat{X}_n\hat{P}_k, \hat{P}_n\right] &= \hat{X}_n\left[\cancel{\hat{P}_k, \hat{P}_n}\right] + \left[\cancel{\hat{X}_n, \hat{P}_n}\right]\hat{P}_k && (25)\\
\left[\hat{X}_n\hat{P}_k, \hat{X}_j\right] &= \hat{X}_n\left[\hat{P}_k, \hat{X}_j\right] + \left[\cancel{\hat{P}_k, \hat{X}_j}\right]\hat{P}_k && (26)\\
\left[\hat{X}_n\hat{P}_k, \hat{X}_j\hat{P}_n\right] &= \hat{X}_j\left[\hat{X}_n, \hat{P}_n\right]\hat{P}_k && (27)
\end{aligned}$$

$$\begin{aligned}
\left[\hat{L}_j, \hat{L}_k\right] &= \hat{X}_k\left[\hat{P}_n, \hat{X}_n\right]\hat{P}_j + \hat{X}_j\left[\hat{X}_n, \hat{P}_n\right]\hat{P}_k && (28)\\
&= \hat{X}_k(-i\hbar)\hat{P}_j + \hat{X}_j(i\hbar)\hat{P}_k && (29)\\
&= i\hbar\left[\hat{X}_j\hat{P}_k - \hat{X}_k\hat{P}_j\right] && (30)\\
&= i\hbar\hat{L}_n && (31)
\end{aligned}$$

Try to solve this part using Levi-Civita symbols and the commutator identities given in page 20 of Shankar.

PROBLEM 29 - Shankar Exercise 10.2.3

The Hamiltonian for the three dimensional isotropic harmonic oscillator is given by

$$\hat{H} = \frac{1}{2m}\left(\hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2\right) + \frac{1}{2}m\omega^2\left(\hat{X}^2 + \hat{Y}^2 + \hat{Z}^2\right)$$

(a) By writing the eigenstates of the Hamiltonian in the *energy* basis as $|n\rangle = |n_x, n_y, n_z\rangle$ (or equivalently $|n\rangle = |n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle$), show that the eigenvalues of the Hamiltonian are given by

$$E(n_x, n_y, n_z) = \left(N + \frac{3}{2}\right)\hbar\omega;\ \text{where } N = n_x + n_y + n_z.$$

$$
\begin{aligned}
\hat{H}\,|n\rangle &= \hat{H}_x\,|n\rangle + \hat{H}_y\,|n\rangle + \hat{H}_z\,|n\rangle && (32)\\
\hat{H}_x\,|n_x, n_y, n_z\rangle &= \hat{H}_x\,|n_x\rangle \otimes \mathbb{1}\,|n_y\rangle \otimes \mathbb{1}\,|n_z\rangle && (33)\\
\hat{H}_y\,|n_x, n_y, n_z\rangle &= \mathbb{1}\,|n_x\rangle \otimes \hat{H}_y\,|n_y\rangle \otimes \mathbb{1}\,|n_z\rangle && (34)\\
\hat{H}_z\,|n_x, n_y, n_z\rangle &= \mathbb{1}\,|n_x\rangle \otimes \mathbb{1}\,|n_y\rangle \otimes \hat{H}_z\,|n_z\rangle && (35)\\
\hat{H}\,|n\rangle &= \sum_i \left[\left(n_i + \frac{1}{2}\right)\hbar\omega\right]|n\rangle && (36)\\
&= \hbar\omega\left[\left(n_x + \frac{1}{2}\right) + \left(n_y + \frac{1}{2}\right) + \left(n_z + \frac{1}{2}\right)\right] && (37)\\
&= \hbar\omega\left(N + \frac{3}{2}\right) && (38)
\end{aligned}
$$

(b) Write the eigenfunctions of the Hamiltonian in terms of single-oscillator wavefunctions in the $|x, y, z\rangle$ basis and verify that the parity of a state with a given N is $(-1)^N$. Moreover, show that the degeneracy of a level with energy $E = \left(N + \frac{3}{2}\right)\hbar\omega$ is $g = (N+1)(N+2)/2$.

$$
\begin{aligned}
\Psi_N(\vec{x}) &= |x, y, z\rangle \langle x, y, z|n_x, n_y, n_z\rangle && (39)\\
&= |x, y, z\rangle \left(\langle x|n_x\rangle \otimes \langle y|n_y\rangle \otimes \langle z|n_z\rangle\right) && (40)\\
\langle x|n_x\rangle &= \frac{1}{(n!)^{1/2}}\left[\frac{1}{2^{1/2}}\left(\gamma - \frac{\mathrm{d}}{\mathrm{d}\gamma}\right)\right]^n \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\gamma^2/2} && (41)\\
\Psi_{ni}(x_i) &= A_{ni} H_{ni}(x_i/b) e^{-x_i^2/2b^2} && (42)\\
\Psi_{ni}(x_i) &= \frac{1}{\left(\pi b^2 2^{2n_i}(n!)^2\right)^{1/4}} H_n(x_i/b) e^{-x_i^2/2b^2}\,|x, y, z\rangle && (43)\\
&= A_{nx} A_{ny} A_{nz} H_{nx}\left(\frac{x}{b}\right) H_{ny}\left(\frac{y}{b}\right) H_{nz}\left(\frac{z}{b}\right) e^{-\frac{1}{2b}(x^2+y^2+z^2)} && (44)
\end{aligned}
$$

$$
= \frac{1}{(\pi b^2)^{3/4}}\left[\frac{e^{-\frac{1}{2b}(x^2+y^2+z^2)}}{\left(2^{2(n_x+n_y+n_z)} n_x! n_y! n_z!\right)^{1/4}}\right] H_{nx}\left(\frac{x}{b}\right) H_{ny}\left(\frac{y}{b}\right) H_{nz}\left(\frac{z}{b}\right) \quad (45)
$$

It can be seen that parity switches due to the Hermite polynomial term by the derivative on the gaussian function $\left(e^{-\gamma^2/2}\right)$. Therefore, $(-1)^{n_x+n_y+n_z} = (-1)^N$

For degeneracy of a level with energy $E = \left(N + \frac{3}{2}\right)\hbar\omega$:

n_x	n_y	options
$n_x = 0$	$n_y = 0, \ldots, n$	$n+1$
$n_x = 1$	$n_y = 0, \ldots, n-1$	n
$n_x = 2$	$n_y = 0, \ldots, n-2$	$n-1$
$\vdots$	$\vdots$	$\vdots$
$n_x = n$	$n_y = 0$	1

(46)

$$\begin{aligned} \sum_{n_x=0}^{n} (N - n_x + 1) &= (N+1) - \frac{N(N+1)}{2} \quad (47) \\ &= \frac{(N+1)(N+2)}{2} \quad (48) \end{aligned}$$

(c) Re-write the first four eigenstates of the Hamiltonian (i.e., the ones with the four lowest energies) in terms of spherical coordinates (r, θ, ϕ) rather than cartesian coordinates (x, y, z).

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$\begin{aligned} \Psi_{000} &= \Psi_0(x)\Psi_0(y)\Psi_0(0) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/(2\hbar)} \quad (49) \\ \Psi_{100} &= \Psi_1(x)\Psi_0(y)\Psi_0(0) = \frac{1}{\sqrt{2\pi}}\left(\frac{m\omega}{\pi\hbar}\right)^{5/4} e^{-m\omega r^2/(2\hbar)} r\sin\theta\cos\phi \quad (50) \\ \Psi_{010} &= \Psi_0(x)\Psi_1(y)\Psi_0(0) = \frac{1}{\sqrt{2\pi}}\left(\frac{m\omega}{\pi\hbar}\right)^{5/4} e^{-m\omega r^2/(2\hbar)} r\sin\theta\sin\phi \quad (51) \\ \Psi_{001} &= \Psi_0(x)\Psi_0(y)\Psi_1(0) = \frac{1}{\sqrt{2\pi}}\left(\frac{m\omega}{\pi\hbar}\right)^{5/4} e^{-m\omega r^2/(2\hbar)} r\cos\theta \quad (52) \end{aligned}$$

(53)

PROBLEM 30 - Shankar Exercise 10.3.3

Imagine a situation in which there are three particles and only three states $|a\rangle$, $|b\rangle$, $|c\rangle$, with energies $\epsilon_a < \epsilon_b < \epsilon_c$. You may assume that the three states form an orthonormal basis.

(a) Show that the total number of allowed, distinct configurations for this system is 27 if all three particles are *distinguishable*.

Since there are three particles which can be in three different states, the particles can be in N^M states or $3^3 = 27$ states.

(b) Show that the total number of allowed, distinct configurations for this system is 10 if all three particles are *identical bosons*. Compute the energy of each of these 10 states and write the normalized first excited state of the system taking into account that it must be *symmetric* under the exchange of any two bosons.

a	b	c	ϵ_{tot}
111			$3\epsilon_a$
	111		$3\epsilon_b$
		111	$3\epsilon_c$
1	11		$\epsilon_a + 2\epsilon_b$
1		11	$\epsilon_a + 2\epsilon_c$
11	1		$2\epsilon_a + \epsilon_b$
	1	11	$\epsilon_b + 2\epsilon_c$
	11	1	$2\epsilon_b + \epsilon_c$
11		1	$2\epsilon_a + \epsilon_c$
1	1	1	$\epsilon_a + \epsilon_b + \epsilon_c$

From the chart, it can be seen there are 10 different states for bosons. Computing for the first excited state of the system:

$$|abc, S\rangle = \frac{1}{\sqrt{3}} \left(|aab\rangle + |aba\rangle + |baa\rangle\right) \tag{54}$$

(c) Show that the total number of allowed, distinct configurations for this system is 1 if all three particles are identical fermions. Compute the energy of this unique state and write the normalized state of the system taking into account that it must be antisymmetric under the exchange of any two fermions.

a	b	c	ϵ_{tot}
1	1	1	$\epsilon_a + \epsilon_b + \epsilon_c$

For the normalized state of the system:

$$|abc, A\rangle = \frac{1}{\sqrt{3!}} \left(|abc\rangle - |acb\rangle + |bca\rangle - |bac\rangle + |cab\rangle - |cba\rangle\right) \tag{55}$$